Recall streamfunction $\psi = \int \underset{\sim}{u} \cdot \underset{\sim}{n} \, ds$

and $\dfrac{\partial \psi}{\partial x} = -v$, $\dfrac{\partial \psi}{\partial y} = u$

and $\underline{\nabla} \cdot \underset{\sim}{u} = 0$ identically.

How is the streamfunction related to vorticity?

Vorticity $\underset{\sim}{\omega} = \underline{\nabla} \times \underset{\sim}{u} = \omega_z \underline{e_z} \equiv \omega \underline{e_z}$

(vorticity is out-of-plane for 2D flows)

and

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = -\nabla^2 \psi$$

In 2D $\boxed{\omega = -\nabla^2 \psi}$

By using ψ, we don't have to solve continuity. What happens to Navier–Stokes?

In general we can formulate the vorticity transport equation by taking $\underline{\nabla} \times$ (Navier Stokes)

This is easiest if we use

$$\underset{\sim}{u} \times \underset{\sim}{\omega} = \underset{\sim}{u} \times (\underline{\nabla} \times \underline{u}) = \frac{1}{2} \underline{\nabla}(\underset{\sim}{u} \cdot \underset{\sim}{u}) - \underline{u} \cdot \underline{\nabla} \underline{u}$$

to write Navier–Stokes as

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + \overbrace{\frac{1}{2} \underline{\nabla}(\underset{\sim}{u} \cdot \underset{\sim}{u}) - \underset{\sim}{u} \times \underset{\sim}{\omega}}^{\underline{u} \cdot \underline{\nabla} \underline{u}} = -\frac{1}{\rho} \underline{\nabla} p + \nu \nabla^2 \underline{u}$$

Now we take the curl and recall that

$$\underline{\nabla} \times \underline{\nabla} \phi = \underline{0}$$

and so

$$\underline{\nabla} \times \frac{\partial \underline{u}}{\partial t} - \underline{\nabla} \times \underline{u} \times \underline{\omega} = \underline{\nabla} \times \nu \nabla^2 \underline{u}$$

$$\Rightarrow \frac{\partial}{\partial t}(\underline{\nabla} \times \underline{u}) - (\underline{\omega} \cdot \underline{\nabla})\underline{u} + (\underline{u} \cdot \underline{\nabla})\underline{\omega} - \underline{u}\, \underbrace{\underline{\nabla} \cdot \underline{\omega}}_{\underline{\nabla} \cdot (\underline{\nabla} \times \underline{u}) = 0}$$
$$+ \underline{\omega}\, \underbrace{\underline{\nabla} \cdot \underline{u}}_{\text{incompressibility}} = \nu \nabla^2 \underline{\nabla} \times \underline{u}$$

Hence

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} + \underline{u} \cdot \underline{\nabla} \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} \underline{u} + \nu \nabla^2 \underline{\omega}}$$

$$\Rightarrow \boxed{\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \underline{\nabla} \underline{u} + \nu \nabla^2 \underline{\omega}}$$

Vorticity Transport equation.

- Expresses the rate of change of angular velocity of fluid particles (LHS)
- Vorticity diffuses via viscous effects. (Final term on RHS)
- $(\underline{\omega} \cdot \underline{\nabla} \underline{u})$ is the vortex stretching term
  velocity gradients lead to increases in vorticity
- $\underline{\omega} \cdot \underline{\nabla} \underline{u} = 0$ in 2D flow because $\underline{\omega}$ is out-of-plane and $\underline{\nabla} \underline{u}$ is in-plane.
- No vortex stretching in 2D.
  $\Rightarrow$ If there is no viscosity, vorticity is conserved.

For low Re we can scale the equation

$$\underline{u} = U\underline{u}^*, \quad \underline{r} = a\underline{r}^*, \quad t = \frac{a}{U}t^*$$

$$\underline{\omega} = \frac{U}{a}\underline{\omega}^*$$

* indicates dimensionless variable.

In index form, vorticity transport equation is:

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2}$$

$$\Rightarrow \frac{U/a}{a/U} \frac{\partial \omega_i^*}{\partial t^*} + \frac{U^2}{a^2} u_j^* \frac{\partial \omega_i^*}{\partial x_j^*} = \frac{U^2}{a^2} \omega_j^* \frac{\partial u_i^*}{\partial x_j^*} + \frac{\nu U}{a^3} \frac{\partial^2 \omega_i^*}{\partial x_j^{*2}}$$

$$\Rightarrow \frac{U^2}{a^2}\left[\frac{D\underline{\omega}^*}{Dt^*} = \underline{\omega}^* \cdot \nabla \underline{u}^*\right] + \frac{\nu U}{a^3} \nabla^2 \underline{\omega}^*$$

In 2D, we obtain

$$\underbrace{\frac{aU}{\nu}}_{Re} \frac{D\underline{\omega}^*}{Dt^*} = \nabla^2 \underline{\omega}^* \quad \Rightarrow \boxed{Re \frac{D\underline{\omega}^*}{Dt^*} = \nabla^2 \underline{\omega}^*}$$

If Re = 0 (Stokes flow) then $\nabla^2 \underline{\omega} = \underline{0}$

[Obtain directly from Stokes

$$\nabla p = \nu \nabla^2 \underline{u} \quad \Rightarrow \underbrace{\nabla \times \nabla p}_{0} = \nabla \times \nabla^2 \underline{u}$$

$$\Rightarrow \nabla^2 \underline{\omega} = \underline{0}. \quad]$$

In 2D Stokes flow, $\underset{\sim}{\omega} = \omega \underset{\sim}{e_z}$

so $\nabla^2 \omega = 0$ and $\omega = -\nabla^2 \psi$

Thus $\nabla^2 \omega = \nabla^2(-\nabla^2 \psi) = -\nabla^4 \psi = 0$

$$\Rightarrow \boxed{\nabla^4 \psi = 0} \quad \text{— biharmonic equation}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

We must solve 1 scalar equation rather than 3 (awkward) equations — but what about the boundary conditions?