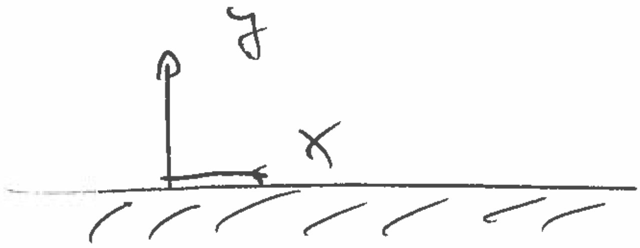


$\longrightarrow U$ for $t > 0$

$u(y,t)$: $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2}$

+ BC & IC.

$$u(y,t) = U \, \mathrm{erfc}\left(\frac{\eta}{2}\right)$$

$$\eta = \frac{y}{\sqrt{\nu t}}$$

See animation on webpage

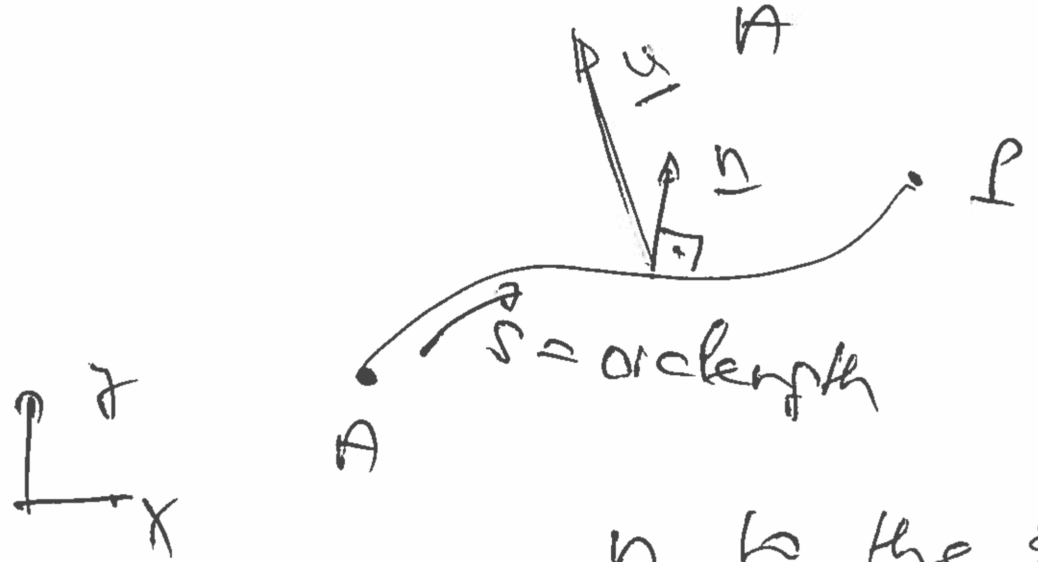
§8 Streamfunction & vorticity eqns

Alternative formulation of N.St. eqns, particularly useful in 2D, incompressible fluids.

Streamfct:

$$\underline{u} = u\,\underline{e_x} + v\,\underline{e_y}$$

Def: $$\psi_A(P) = \int_A^P \underline{u}\cdot\underline{n}\,ds$$



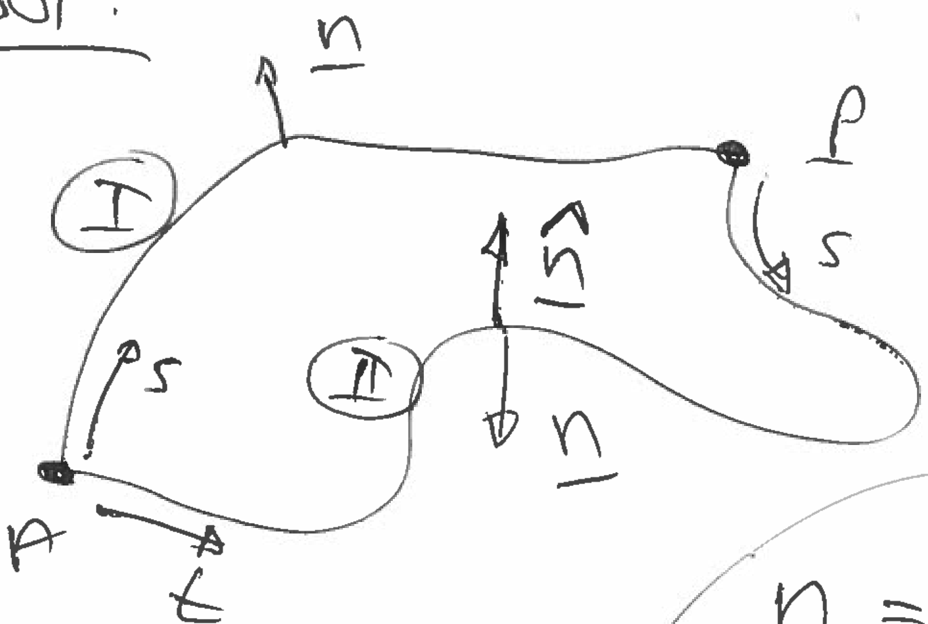
$\underline{n}$ to the left

$\psi_A(P)$ is the vol. flux (per unit depth in the z-direction) crossing the line AP.

Implications:

(1) $\psi_A(P)$ is path independent!

Proof:



$$\underline{n} = -\hat{\underline{n}}$$
$$ds = -dt$$

$$\psi_A^{I}(P) = \int_{A,\,(I)}^{P} \underline{u}\cdot\underline{n}\,ds$$

$$\psi_A^{II}(P) = \int_{A,\,(II)}^{P} \underline{u}\cdot\underbrace{\hat{\underline{n}}}_{-\underline{n}}\,\underbrace{dt}_{-ds}$$

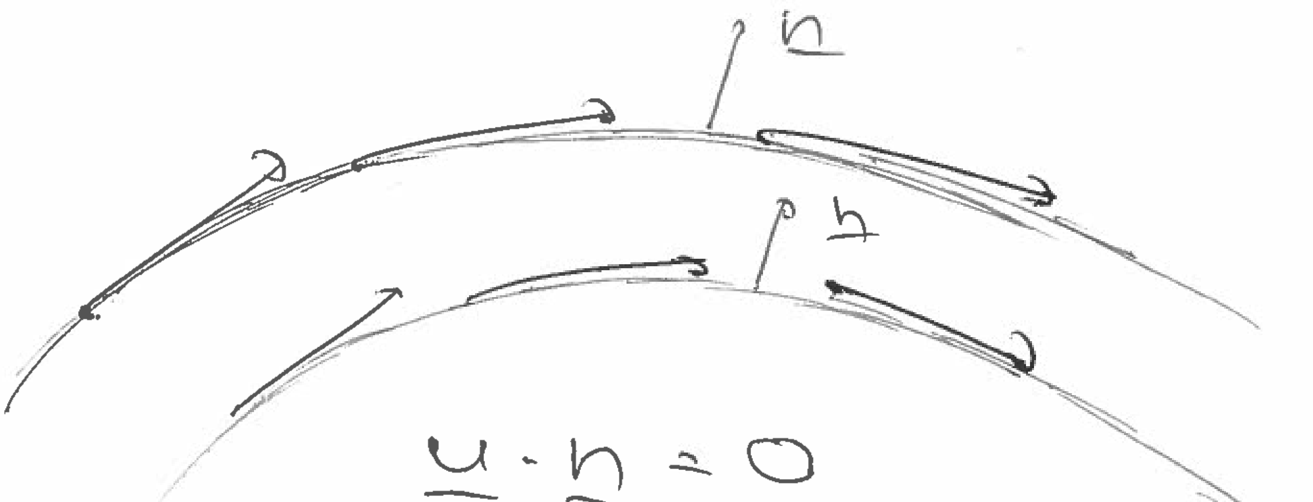
$$= -\int_{P,\,(II)}^{A} \underline{u}\cdot\underline{n}\,ds$$

(4)

$$\psi_A^{I}(P) - \psi_A^{II}(P) = \int_{A,\,\mathrm{I}}^{P} \underline{u}\cdot\underline{n}\,ds + \int_{P,\,\mathrm{II}}^{A} \underline{u}\cdot\underline{n}\,ds$$

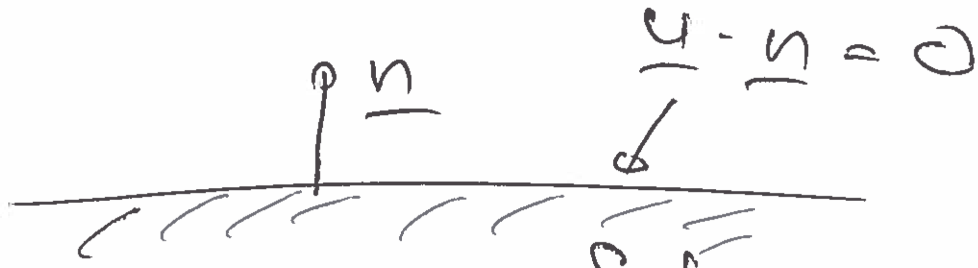
$$= \underbrace{\oint_{APA} \underline{u}\cdot\underline{n}\,ds = 0}_{\text{integral cont. eqn. for incompr. fluid}}$$

(2) ψ is constant along streamlines = lines that are tangential to the veloc field:



$\underline{u}\cdot\underline{n} = 0$

(3) impermeable boundaries are streamlines (in this sense)



<u>Convention:</u> set $\psi = 0$ along such boundaries

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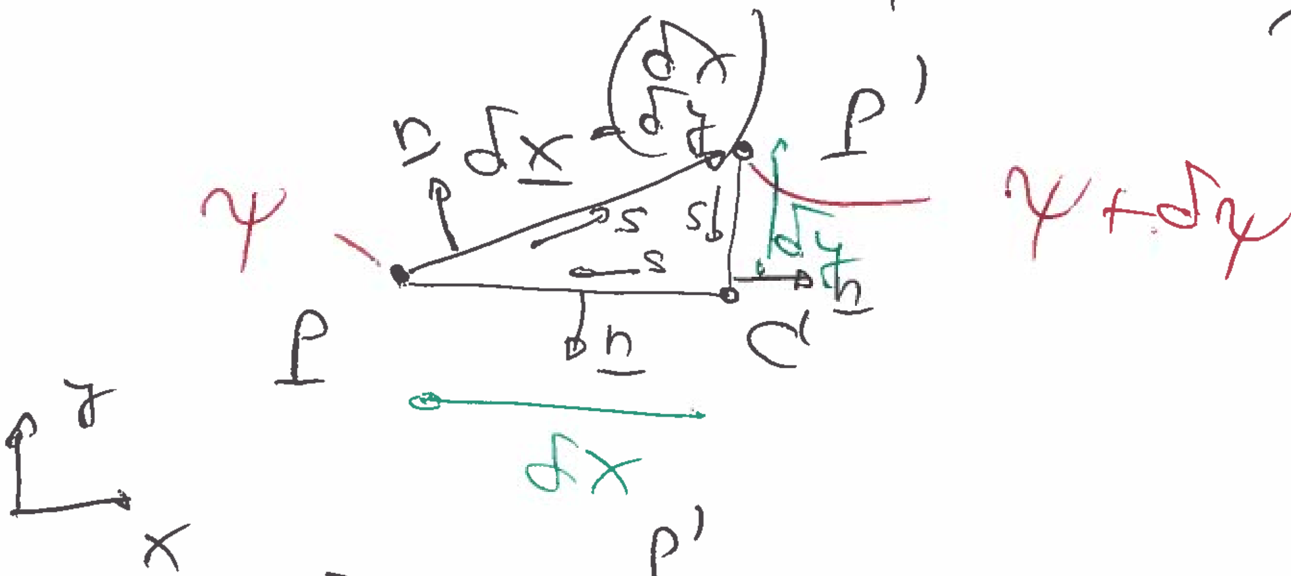
what PDE does ψ satisfy?



$$\delta\psi = \int_P^{P'} \underline{u} \cdot \underline{n}\, ds$$

integral continuity:

$$\oint_{\triangle} \underline{u}\cdot\underline{n}\,ds = 0$$

$$= \underbrace{\int_{f}^{f'} \cdots}_{\delta\psi} + \int_{f'}^{c} \cdots + \int_{c}^{f} \cdots$$

$$\delta\psi = \int_{f}^{f'} \underline{u}\cdot\underline{n}\,ds = -\int_{f'}^{c} \underbrace{\underline{u}}_{u}\cdot\underbrace{\underline{n}\,ds}_{-dy} - \int_{c}^{f} \underbrace{\underline{u}\cdot\underline{n}\,ds}_{(-v)(-dx)}$$

$$\delta\psi = \int_{f'}^{c} u\,dy - \int_{c}^{f} v\,dx$$

Now use mean value theorem & let $\delta x, \delta y \rightarrow 0$

$$\delta\psi = u\,\delta y - v\,\delta x$$

<u>Also:</u> $\psi = \psi(x,y)$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

(2D Taylor expansion)

$$\boxed{\begin{aligned} \frac{\partial \psi}{\partial x} &= -v \\ \frac{\partial \psi}{\partial y} &= u \end{aligned}}$$

Similar to potential, Cauchy-Riemann, Airy stress fcts etc.

<u>Remarks:</u>

(1) Derivation involved continuity eqn.
$\Rightarrow$ maybe the continuity eqn. is

already satisfied:
Is it?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} \stackrel{?}{=} 0$$

$\Rightarrow$ if we can formulate our problems in terms of the stream fct. we can ignore the continuity eqn.