$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

$$\frac{\partial u_k}{\partial x_k} = 0 \qquad + BC$$

$$\left.\begin{aligned} x_i &= a\tilde{x}_i \\ u_i &= U\tilde{u}_i \\ t &= \frac{a}{U}\tilde{t} \\ p &= \frac{\mu U}{a}\tilde{p} \end{aligned}\right\}$$

Param:

a, U, μ, ρ

Non-dimensionalise

$$\boxed{Re \frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \nabla^2 \tilde{u}_i \qquad \frac{\partial \tilde{u}_k}{\partial \tilde{x}_k} = 0 \qquad + BC}$$

$$\boxed{Re = \frac{a\rho U}{\mu}}$$

Alternative:

$$p = \rho U^2 \tilde{p}$$

Simplification of the scaled equations

- Writing the Navier Stokes equations in dimensionless form not only reduces the number free parameters, it also shows the appropriate limiting form of the equations if the Reynolds number approaches extreme values.
- For $Re \to 0$ (slow viscous flow), the viscous pressure scaling is appropriate. Performing the limit $Re \to 0$ in
$$Re\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \tilde{\nabla}^2 \tilde{u}_i$$
yields the *Stokes equations*:
$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \tilde{\nabla}^2 \tilde{u}_i$$
which are linear since the non-linear inertial terms disappear.
- For $Re \to \infty$ (high speed flows), the inertial pressure scaling is appropriate. Performing the limit $Re \to \infty$ in
$$\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\tilde{\nabla}^2 \tilde{u}_i$$
shows that such flows are governed by the *Euler equations*
$$\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i}$$
- Note that the order of the Euler equations is lower than that of the full Navier Stokes equations (first rather than second spatial derivatives!). This means that not all boundary conditions can be applied on the surface of solid bodies.
- Typically, the no-slip condition is discarded in favour of the no-penetration condition (compare to inviscid flow theory which is also governed by these equations – in fact, the Euler equations can be derived by setting the viscosity to zero).

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- However, close to the surface of the body, the no-slip condition always becomes important since viscosity (no matter how small) will always reduce the fluid velocity to zero as one approaches the surface of the solid body. This manifests itself in the existence of a thin layer (a so-called boundary layer) in which viscous effects are important and in which the velocity varies rapidly to fulfill the no-slip condition.

- Mathematically, the limit $Re \to \infty$ represents a *singular limit* and the solution has to be found by matched asymptotic expansions.

- We will briefly look at boundary layers at the end of this course.

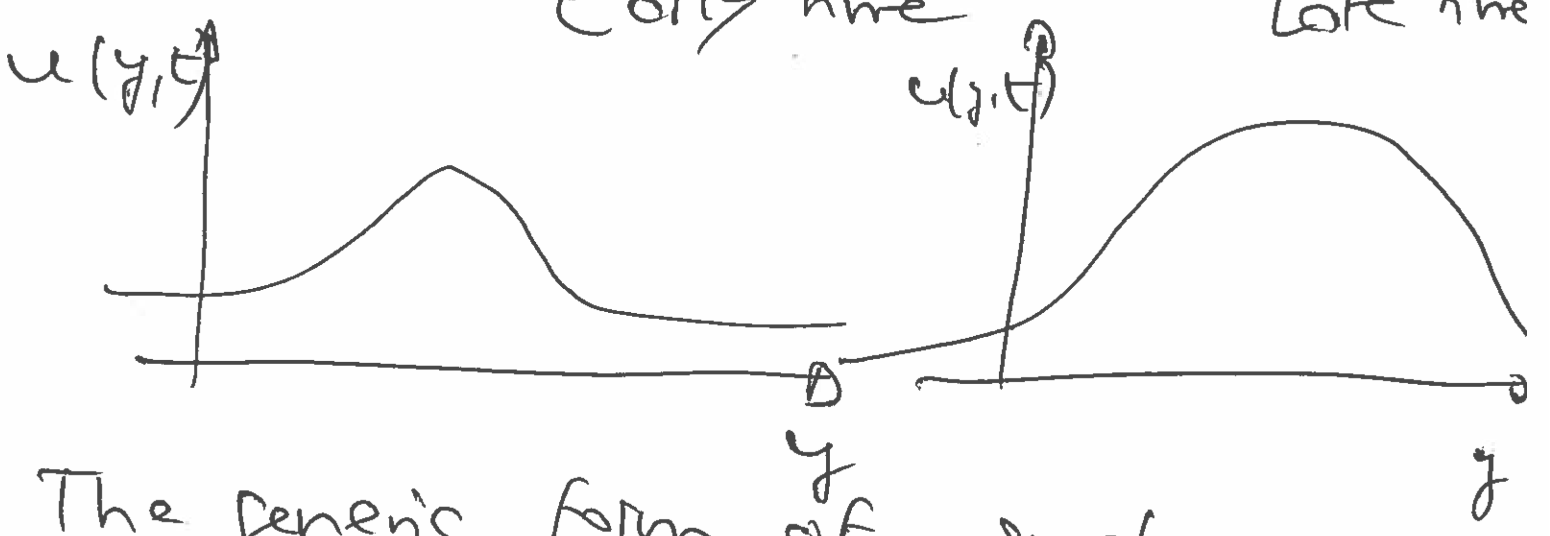
Further comments

- The choice of the 'right' scales often requires some physical intuition. Especially when we use scaling arguments to simplify the equations (by dropping small terms), we have to choose the scales for the physical quantities such that the non-dimensional quantities are all of comparable magnitude ('of order one').

§... Similarity solutions (4

often solutions of PDEs are self-similar, i.e. they have the same "shape" but possibly different scales at different times.

E.g.:



The generic form of such solutions is

$$\boxed{u(y,t) = a(t)\, f\left(\frac{y}{b(t)}\right)}$$

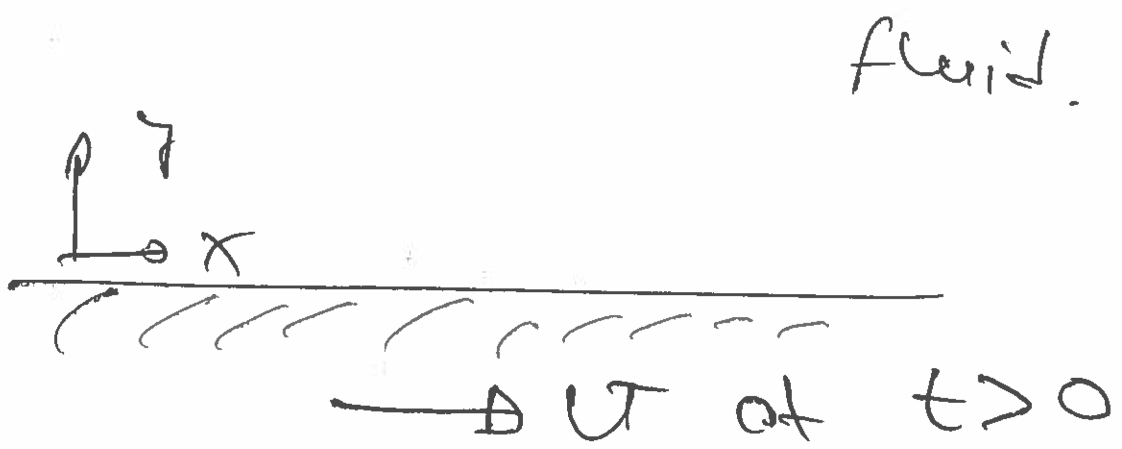
This scales the amplitude

This scales the width of the soln

- Sim. solns. don't always exist – try it!
- They often reduce PDEs to ODEs for $f(\eta)$ where
$$\eta = \frac{y}{b(t)}$$ is the similarity variable
- The existence of sim. solns is often suggested or even implied by dimensional arguments.

Example:

Rayleigh's jerked plate



Assume parallel flow:

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \text{for } u(y,t) \\ &\text{IC: } u(y, t=0) = 0 \\ &\text{BC: } \quad u|_{y=0} = U \;(*) \quad \text{for } t > 0 \\ &\qquad u \to 0 \quad \text{as} \quad y \to \infty \end{aligned}}$$

Want: $u(y, t; \nu, U)$

Note: PDE is linear & homog.

BC & IC are linear in u

& homogeneous apart from (*).

$\Rightarrow$ $u(y, t; \nu, U)$ must be linear in U:

$$u(y, t; \nu, U) = U f(y, t; \nu)$$

Proof: Assume $\hat{u}$ is a soln of the eqns for given value of U:

$$\frac{\partial 2\hat{u}}{\partial t} = \nu \frac{\partial^2 2\hat{u}}{\partial y^2}$$

$$2\hat{u}(y, t=0) = 0$$

$$2\hat{u}\Big|_{y=0} = 2U$$

$$2\hat{u} \to 0 \quad \text{as } y \to \infty$$

Now solve the following problem:

$$\frac{\partial \hat{\hat{u}}}{\partial t} = \nu \frac{\partial^2 \hat{\hat{u}}}{\partial y^2}$$

$$\hat{\hat{u}}(y, t=0) = 0$$

$$\hat{\hat{u}}\Big|_{y=0} = 2U$$

$$\hat{\hat{u}} \to 0 \quad \text{as } y \to \infty$$

Claim: $\hat{\hat{u}} = 2\hat{u}$

q.e.d.

$$u(y,t;\nu,U) = U\, f(y,t;\nu)$$

Now check dimensions

$$[u] = \frac{m}{sec}$$

$$[U] = \frac{m}{sec}$$

$\Rightarrow$ f has to be a non-dimensional fct. of its arguments.

$$[y] = m$$

$$[t] = sec$$

$$[\nu] = \frac{m^2}{sec}.$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}\ ; \quad [\nu] = \frac{\left[\frac{\partial u}{\partial t}\right]}{\left[\frac{\partial^2 u}{\partial y^2}\right]}$$

$$[\nu] = \frac{\left(\frac{m}{sec^2}\right)}{\left(\frac{\cancel{m}}{sec\, m^{\cancel{2}}}\right)} = \frac{m^2}{sec}$$

Sim. variable

$$\eta = \frac{\nu t}{y^2}$$

or $$\eta = \frac{\sqrt{\nu t}}{y}$$

or $$\eta = \sin\left(\frac{\nu^4 t^4}{y^8}\right)$$

or $\vdots$

but we wanted a choice of η that was linear in y:

$$\eta = \frac{y}{\sqrt{\nu t}}$$

$b(t) = \sqrt{\nu t}$

soln <u>must</u> have the form

$$u(y,t;\nu,U) = U f\Big(\underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta}\Big)$$

$$= U f\big(\underbrace{y\,(\nu t)^{-1/2}}\big)$$

$$\eta = y\,(\nu t)^{-1/2}$$

Into PDE & BCs:

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t}$$

$$\frac{\partial u}{\partial t} = U f' \frac{y}{\sqrt{\nu}}\left(-\frac{1}{2}\right) t^{-3/2}$$

$$\frac{\partial u}{\partial y} = U \frac{df}{d\eta}\underbrace{\frac{\partial \eta}{\partial y}}_{\frac{1}{\sqrt{\nu t}}}$$

$$\frac{\partial^2 u}{\partial y^2} = U f'' \frac{1}{\nu t}$$

into PDE:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$-\frac{1}{2}\cancel{U}\underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta}\cancel{\frac{1}{t}} f' = \cancel{\nu}\cancel{U} f'' \frac{1}{\cancel{\nu t}}$$

$$\boxed{f'' + \frac{1}{2}\eta f' = 0}$$

2^{nd} order ODE for $f(\eta)$.

$u = U f(\underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta})$

BC:

$u = U$ at $y = 0$

$$\boxed{f(\eta = 0) = 1}$$

$u \to 0$ as $y \to \infty$

$$\boxed{f \to 0 \text{ as } \eta \to \infty}$$

IC: $u = 0$ at $t = 0$

$u \to 0$ as $t \to 0$

$\cancel{f = 0}$

$$\boxed{f \to 0 \quad \text{as} \quad \eta \to \infty}$$

3 cond. for 2^{nd} order ODE

but 2 cond. are identical

Solve:

$$f'' + \frac{1}{2}\eta f' = 0$$

Subst:

$$f' = F$$

$$F' + \frac{1}{2}\eta F = 0$$

$$\frac{dF}{d\eta} = -\frac{1}{2}\eta F$$

$$\int \frac{1}{F}\, dF = \int -\frac{1}{2}\eta\, d\eta$$

$$\ln\left(\frac{F}{F_0}\right) = -\frac{1}{4}\eta^2$$

F_0 → const.

$$F = f' = f_0 \exp\left(-\frac{1}{4}\eta^2\right)$$

$$f(\eta) = A + f_0 \int^{\eta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

what about lower limit of integral? Arbitrary!
Choose ∞!

$$f(\eta) = A + f_0 \int_{\infty}^{\eta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

$$f(\eta) = A + B \int_{\eta}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

Apply BC:

$f \to 0$ as $\eta \to \infty$

$\Rightarrow A = 0$

$$f(\eta = 0) = 1 = B \underbrace{\int_0^\infty \exp\left(-\tfrac{1}{4}\xi^2\right) d\xi}_{\sqrt{\pi}}$$

$$u = \frac{U}{\sqrt{\pi}} \int_\eta^\infty \exp\left(-\tfrac{1}{4}\xi^2\right) d\xi$$

where $\eta = \dfrac{y}{\sqrt{\nu t}}$

$$u = U \operatorname{erfc}\left(\frac{\eta}{2}\right)$$