DIMENSIONAL ANALYSIS AND SCALING

Observation 1:

- Consider the flow past a sphere:

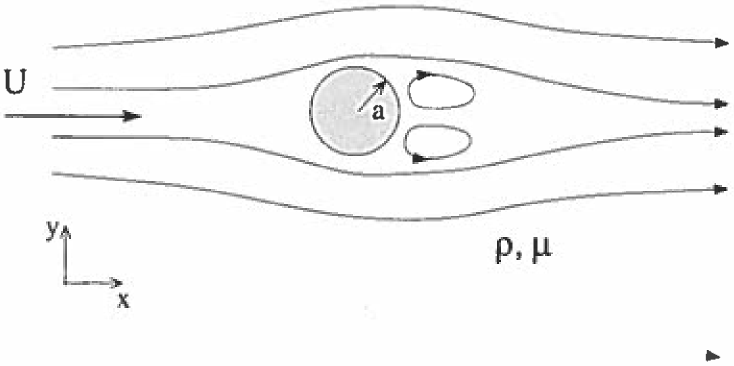


Figure 1: Flow past a sphere. Far away from the sphere of radius a, the fluid has a uniform velocity, $\mathbf{u} = U\mathbf{e}_x$.

- To determine the velocity field we need to solve the Navier-Stokes equations

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i$$

together with the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0,$$

subject to the boundary conditions

$u_i = 0$ for $r = a$ (no slip on the surface of the sphere)

and

$\mathbf{u} \to U\mathbf{e}_x$ as $r \to \infty$ (uniform velocity far away from the sphere).

- The solution which we could obtain by solving the equations analytically, numerically or even by carrying out experiments (!) will have the form

$$\mathbf{u} = \mathbf{u}(x, y, z, t; \rho, \mu, a, U),$$

  i.e. the velocity field will depend on the spatial coordinates, on time and on the four physical parameters appearing in the problem.

- This implies that a change to any one of the physical parameters will (in general) change the entire flow field.
- This is not a problem if we can find an exact analytical solution which explicitly shows the dependence of the solution on each parameter.
- However, if we perform numerical computations (in which all parameters have to be given fixed numerical values), then each change of a physical quantity would require a completely new computation.
- If we perform experiments, then a different experiment has to be performed for each set of physical parameters (such as doubling the size of the sphere, making the fluid more viscous, etc.).
- Think of the implications for (e.g.) wind tunnel testing. If the above was true, then to obtain the flow field past a newly developed prototype car, you'd have to build the car in its full size. This might not be a problem but what about testing jumbo jets...?

Observation 2:

- When we solve the Navier Stokes equations (or any other equation of continuum mechanics), we tend to get results like

$$u = \sin(r).$$

- Do we really?
- What about the dimensions of the above equation?

$$\underbrace{u}_{\text{m/sec}} = \sin(\underbrace{r}_{m}).$$

- How do you take the sin of 'metres'?
- Actually, we tend to get results like

$$u/U = \sin(r/a),$$

  i.e. all quantities appear in dimensionless form.

- The fact that the equations of continuum mechanics are derived from (dimensionally coherent!) physical statements implies that we can *always* write our equations in dimensionless form.

Non-dimensionalisation

- We obtain non-dimensional equations by non-dimensionalising all quantities with characteristic scales which are in the problem. E.g.

$$\underbrace{\mathbf{u}}_{\text{dimensional velocity}} = \underbrace{U}_{\substack{\text{velocity scale:}\\ \text{velocity far from}\\ \text{the sphere}}} \underbrace{\tilde{\mathbf{u}}}_{\text{non-dimensional velocity}}$$

- Convention: Use a tilde to distinguish dimensional from non-dimensional variables (where necessary).
- The non-dimensionalisation typically reduces the number of free parameters in the problem and shows that 'similar' physical problems often have 'similar' solutions.
- A very useful side-effect of the non-dimensionalisation is that the non-dimensionalised equations provide additional insight into the relative size of the various terms in the equations (provided the 'scales' were chosen appropriately).
- The identification of small terms in an equation often motivates significant simplifications which can be obtained by neglecting the small terms against bigger ones.

Example: Flow past sphere

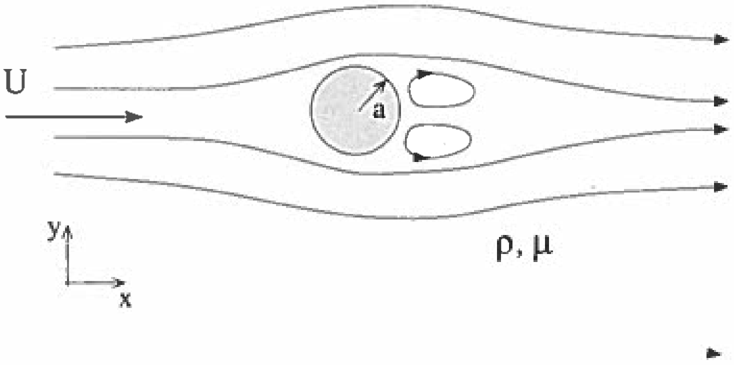


Figure 2: Flow past a sphere. Far away from the sphere of radius a, the fluid has a uniform velocity, $\mathbf{u} = U\mathbf{e}_x$.

- Scales:

  length scale: a

  velocity scale: U

  time scale: Steady boundary conditions, so there's no explicit time scale in the problem. Hence, we need to construct a time scale from the available parameters. Choose: $T = a/U$.

  pressure scale: There's no natural scale for the pressure. We can construct two reference pressures from the physical parameters.

  - $P = \rho U^2$ which is a dynamic pressure. This is appropriate if we expect dynamic effects to be dominant, i.e. for high velocity flows

  or

  - $P = \mu U/a$ which is a viscous pressure scale. This is appropriate if we expect viscous effects to be dominant, i.e. for slow flows with large viscosity.

- Use these scales to non-dimensionalise the physical quantities:

$$u_i = U\tilde{u}_i$$

$$x_i = a\tilde{x}_i$$

$$t = \frac{a}{U}\tilde{t}$$

$$p = \begin{cases} \rho U^2 \tilde{p} & \text{for the dynamic pressure scale} \\ \mu U/a\, \tilde{p} & \text{for the viscous pressure scale} \end{cases}$$

- Inserting the scaled quantities into the Navier Stokes equations turns the problem of the flow past a sphere into

$$\begin{cases} Re\frac{D\tilde{u}_i}{Dt} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \tilde{\nabla}^2 \tilde{u}_i & \text{for } p = \mu U/a\, \tilde{p} \\ \frac{D\tilde{u}_i}{Dt} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\tilde{\nabla}^2 \tilde{u}_i & \text{for } p = \rho U^2\, \tilde{p} \end{cases}$$

  together with the continuity equation

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_i} = 0,$$

  and the boundary conditions

$$\tilde{u}_i = 0 \quad \text{for } \tilde{r} = 1 \text{ (no slip on the surface of the sphere)}$$

  and

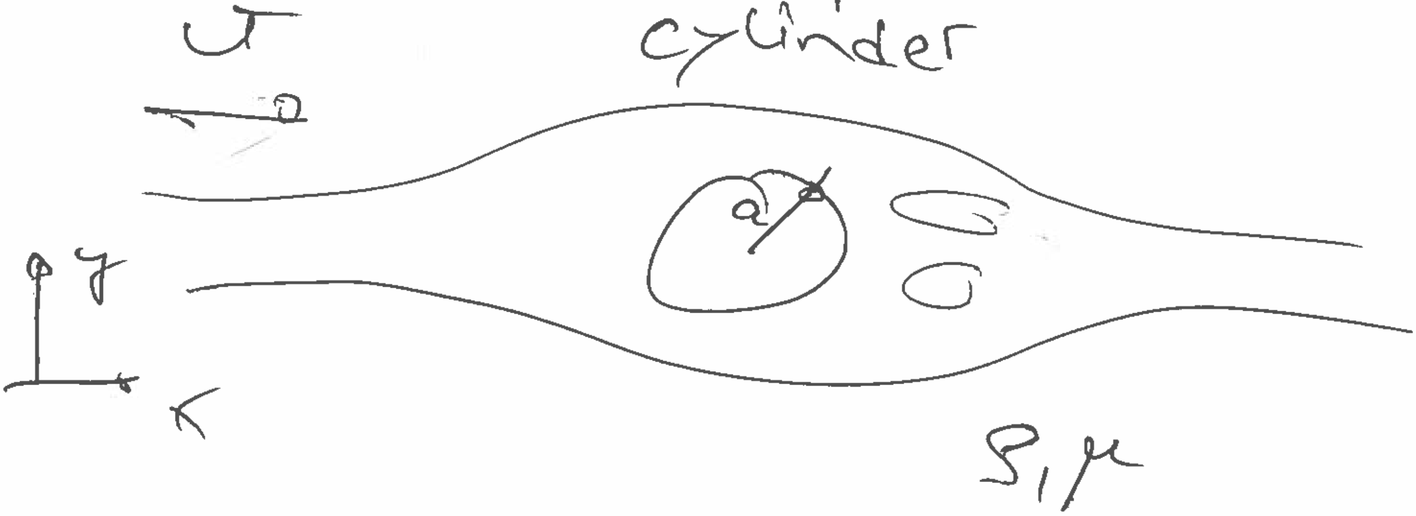
$$\tilde{\mathbf{u}} \to \mathbf{e}_x \quad \text{as } \tilde{r} \to \infty \text{ (uniform velocity far away from the sphere).}$$

- In non-dimensional form, the problem depends only on *one* dimensionless parameter, the *Reynolds number*

$$Re = \frac{\rho a U}{\mu} = \frac{aU}{\nu}$$

  which represents the ratio of inertial to viscous forces in the flow.

Flow past cylinder



- Scales for nondimensionalisation
  - length: radius a
  - veloc: U
  - time: $\frac{a}{U}$ $\qquad \frac{m}{m/sec} = sec$
  - pressure:

    two choices: ρU^2 (inertial)

    $\frac{\mu U}{a}$ (viscous)

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$$u_i = U\tilde{u}_i$$

$$x_i = a\tilde{x}_i$$

$$t = \frac{a}{U}\tilde{t}$$

$$p = \frac{\mu U}{a}\tilde{p}$$

$$\rho\left(\frac{U^2}{a}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \frac{U^2}{a}\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j}\right) =$$

$$= -\frac{\mu U}{a^2}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \mu\frac{U}{a^2}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j \partial \tilde{x}_j}$$

$$\frac{\rho U^2}{a}\frac{D\tilde{u}_i}{D\tilde{t}} = \frac{\mu U}{a^2}\left(-\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j}\right)$$

$$\boxed{\underbrace{\frac{\rho U a}{\mu}}\frac{D\tilde{u}_i}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}}$$

Re = Reynolds number

= ratio of inertial to viscous forces

$$\frac{U}{a}\frac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = 0$$

$$\boxed{\frac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = 0}$$

BC: $U\underline{\tilde{u}} \to U\underline{e_x}$ as $a\tilde{r} \to \infty$

$$\boxed{\underline{\tilde{u}} \to \underline{e_x} \quad \text{as} \quad \tilde{r} \to \infty}$$

$U\underline{\tilde{u}} = \underline{0}$ on $a\tilde{r} = a$

$$\boxed{\underline{\tilde{u}} = \underline{0} \quad \text{on} \quad \tilde{r} = 1}$$

- Hence, in non-dimensional terms, the solution will only depend on one parameter, i.e.

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}; Re).$$

- This implies that the non-dimensional velocity (the ratio of the actual velocity to the velocity far away from the sphere) at a fixed non-dimensional position (e.g. two diameters in front of the sphere) will have the same value for all physical realisations of the experiment provided the Reynolds number of the flows is the same.

- This means that an experiment with a 1:100 scale model of a jumbo jet will give the correct non-dimensional flow field, provided the velocity of the oncoming flow increased by a factor of 100 – and provided any physical effects which are not included in the incompressible Navier Stokes equations are unimportant. [The latter point is important in aerodynamics where compressibility often becomes an issue. Compressibility introduces another non-dimensional parameter (the Mach number) whose value also has to be conserved].

$$Re = \frac{\rho a U}{\mu}$$

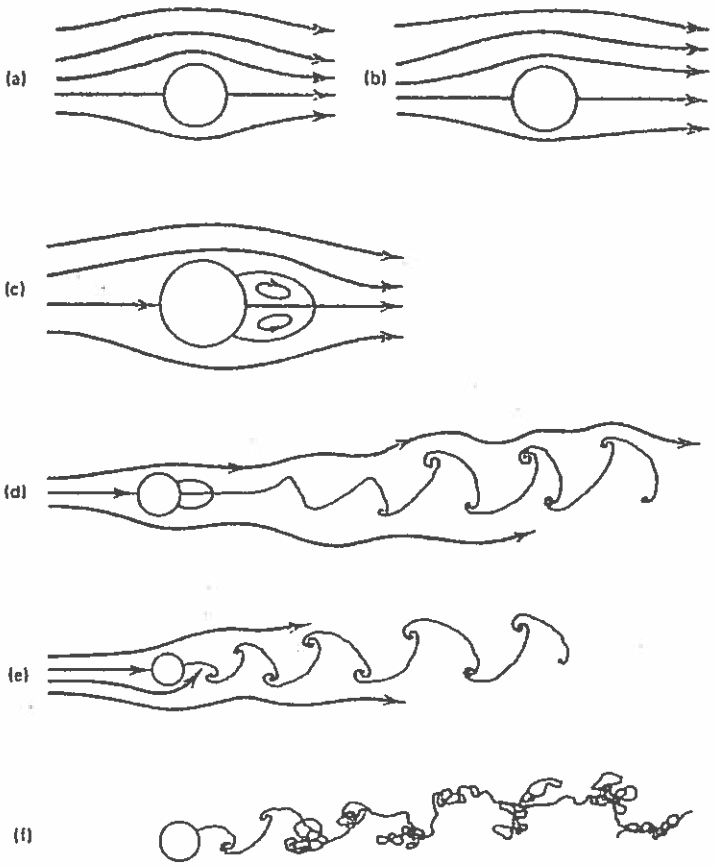


Figure 14.15 Flow regimes for a cylinder: (a) $Re = 0$, symmetrical; (b) $0 < Re < 4$; (c) $4 < Re < 40$, attached vortices; (d) $40 < Re < 60$–100, von Kármán vortex street; (e) 60–$100 < Re < 200$, alternate shedding; (f) $200 < Re < 400$, vortices unstable to spanwise bending.

then shed. Depending on the details of the experiment, this first occurs at a Reynolds number somewhere between 60 and 100. Figure 14.17a shows the vortex street development. As one goes further downstream the circular motion of the vortices is stopped by viscous forces. In an experiment such as Fig. 14.17 it is difficult to see when this happens as the flow visualization marker retains its distinctive pattern even after the vortices have stopped. The first picture in Fig. 14.18 shows a vortex pattern, the same as in Fig. 14.17, extending a distance of 200 diameters behind the (very small) cylinder. The path line streaks in the figure were produced by smoke from vaporizing oil on a hot wire located at the cylinder station. After some downstream [illegible] the vortices have

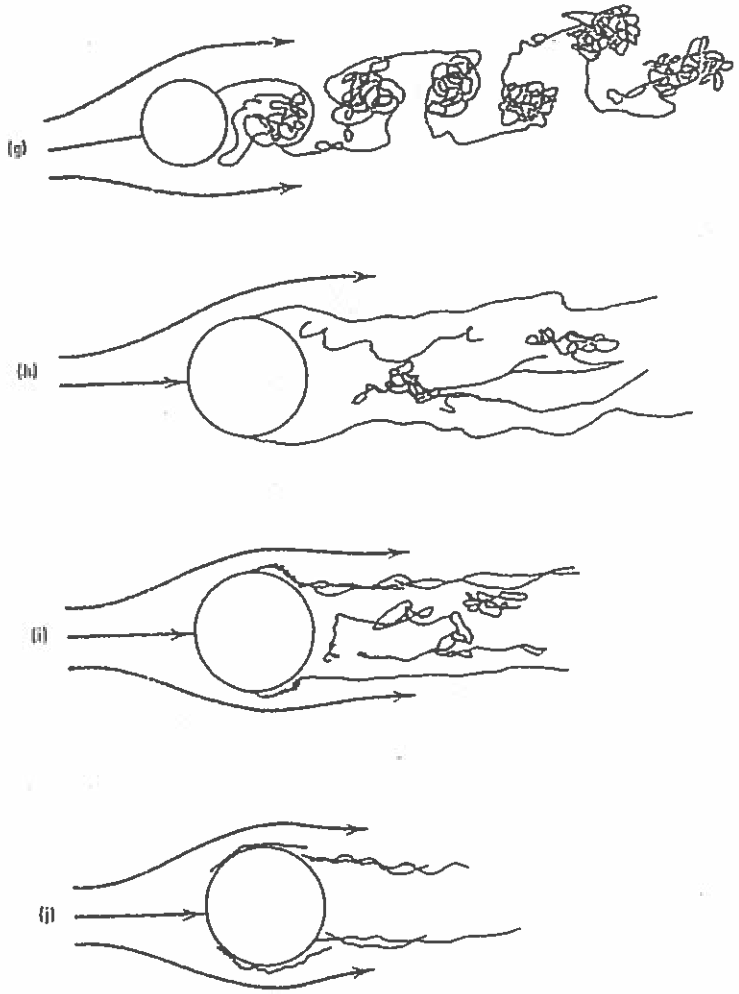


Figure 14.15 (*continued*) (g) $400 < Re$, vortices turbulent at birth; (h) $Re < 3 \times 10^5$, laminar boundary layer separates at 80°; (i) $3 \times 10^5 < Re < 3 \times 10^6$, separated region becomes turbulent, reattaches, and separates again at 120°; (j) $3 \times 10^6 < Re$ turbulent boundary layer begins on front and separates on back.

a location 150 diameters downstream from the cylinder and no vortices exist. The earlier patterns in the picture in Fig. 14.18a are fossils of events that occur where the smoke was introduced. Cimbala et al. (1988) have not only shown the vortex street decay, but they have also vividly demonstrated how our eyes can be deceived by inactive flow visualization patterns.

[illegible] the drag force

Flow past a circular cylinder (Panton p.387)

14.6 FLOW AROUND A CIRCULAR CYLINDER 387

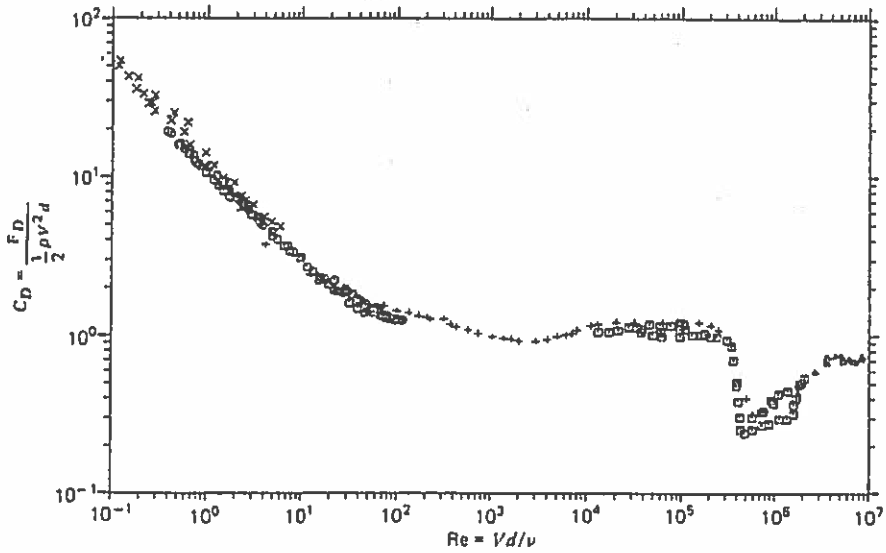
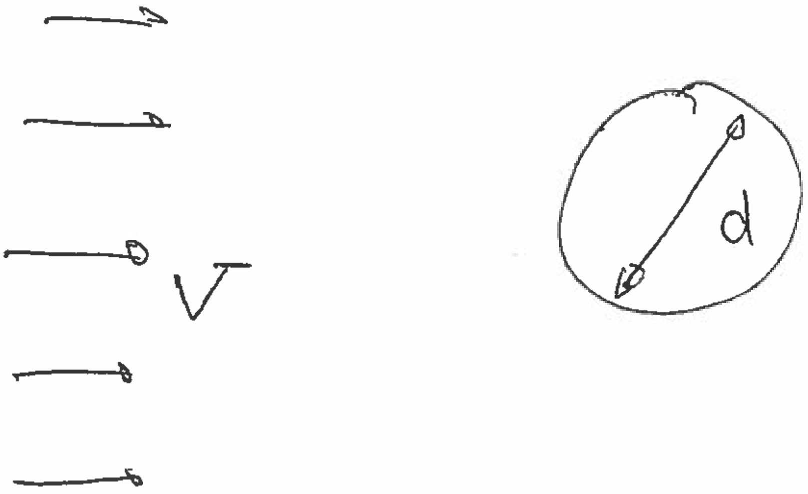


Figure 14.14 Drag curve for a cylinder. Data is from Delany and Sorenson (1953), Finn (1953), Roshko (1961), Tritton (1959), and Wieselsberger (1921).

Data from all kinds of exp. collapses onto the same curve i.e.

$C_D = f(Re)$



Drag force F_D

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 d} \qquad Re = \frac{Vd}{\nu}$$