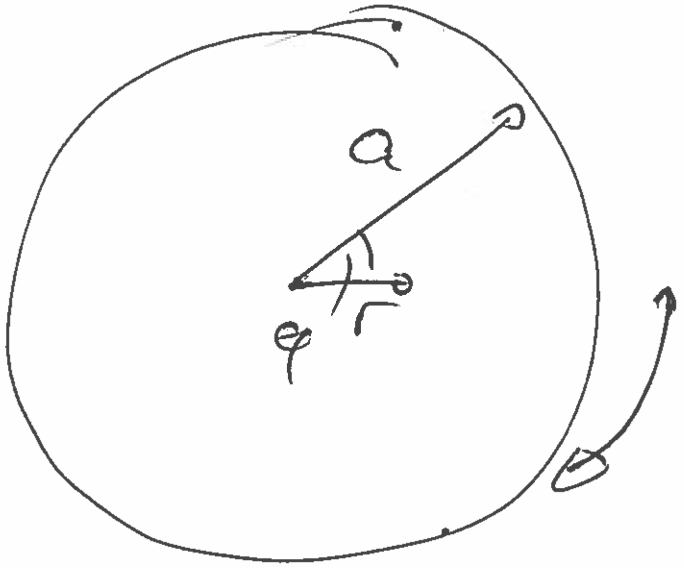
Example: Oscillating cylinder



$$\underline{u} = \frac{\Omega a}{\omega} \cos(\Omega t)\, \underline{e_\varphi}$$

$$\underline{u} = U \cos(\Omega t)\, \underline{e_\varphi}$$

Assumptions:

$$u = u_r = 0$$

$$v = u_\varphi = v(r,t)$$

$$p = p(r,t)$$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} \boxed{- \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\frac{\partial v}{\partial t} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = -\cancel{\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \cancel{\frac{2}{r^2}\frac{\partial u}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = -\cancel{\frac{1}{\rho}\frac{\partial P}{\partial z}} + \cancel{\nu\nabla^2 w},$$

$0 = 0$

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

$0 = 0$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

φ-mom. eqn:

$$\frac{\partial v}{\partial t} = \nu \Big(\underbrace{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}}_{\nabla^2 v(r,t)} - \frac{v}{r^2} \Big) \quad (*)$$

IVP need IC

azimuthal

or
search for
time-periodic soln.

$$v(r,t) = \mathrm{Re}\left(V(r)\, e^{i\Omega t} \right)$$

into $(*)$

$$i\Omega V(r) \cancel{e^{i\Omega t}} = \nu \cancel{e^{i\Omega t}} \left(V''(r) + \frac{1}{r} V'(r) - \frac{V(r)}{r^2} \right)$$

$$\boxed{r^2 V'' + r V' + \left(-\frac{i\Omega}{\nu} r^2 - 1 \right) V = 0} \quad (**)$$

compare

$$\xi^2 f''(\xi) + \xi f'(\xi) + (\xi^2 - n^2) f(\xi) = 0$$

is a Bessel ODE.

coordinate transformation:

$n=1$

$$\xi^2 = -\frac{i\Omega}{\nu} r^2$$

$$r = \sqrt{\frac{i\nu}{\Omega}}\,\xi$$

Now note that derivs in (**) are of the form $r^n \frac{d^n f}{dr^n}$.
Such expressions are invariant under re-scalings: $r \to \alpha r$ ← constant

$$r = \underbrace{\sqrt{\frac{i\nu}{\Omega}}}_{\alpha}\,\xi \qquad \frac{dr}{d\xi} = \alpha$$

$$\frac{df}{dr} = \frac{1}{\left(\frac{dr}{d\xi}\right)} \frac{df}{d\xi} = \frac{1}{\alpha}\frac{df}{d\xi}$$

$$\boxed{r\frac{df}{dr}} = \cancel{\alpha}\,\xi\,\frac{1}{\cancel{\alpha}}\frac{df}{d\xi} = \boxed{\xi\frac{df}{d\xi}}$$

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$$r^2 \frac{d^2 f}{dr^2} = \alpha^2 \xi^2 \frac{d}{dr}\left(\frac{df}{dr}\right)$$

$$= \cancel{\alpha^2} \xi^2 \frac{1}{\cancel{\alpha}} \frac{d}{d\xi}\left(\frac{1}{\cancel{\alpha}} \frac{df}{d\xi}\right)$$

$$r^2 \frac{d^2 f}{dr^2} = \xi^2 \frac{d^2 f}{d\xi^2}$$

So with $r = \sqrt{\frac{i\nu}{\Omega}}\, \xi$ (~~*~~A)

becomes:

$$\xi^2 V''(\xi) + \xi V'(\xi) + (\xi^2 - 1) V(\xi) = 0$$

Bessel ODE with $n = 1$.

Gen. soln. of this ODE is

(lin., 2nd order, homogeneous)

$$f(\xi) = A J_1(\xi) + B Y_1(\xi)$$

Bessel fcts of the 1st & 2nd kind of order 1

So gen. soln of (**):

$$V(r) = A J_1\left(\sqrt{\frac{\Omega}{i\nu}}\, r\right) + B Y_1\left(\sqrt{\frac{\Omega}{i\nu}}\, r\right)$$

BC: $V(r=a) = U$

V finite as $r \to 0$.

Note: $Y_1(\xi) \to \infty$ as $\xi \to 0$

$\Rightarrow B = 0$

$$V(r) = \frac{U}{J_1\left(\sqrt{\frac{\Omega}{i\nu}}\, a\right)} J_1\left(\sqrt{\frac{\Omega}{i\nu}}\, r\right)$$

$$u(r,t) = \mathrm{Re}\left(U \frac{J_1\left(\sqrt{\frac{\Omega}{i\nu}}\, r\right)}{J_1\left(\sqrt{\frac{\Omega}{i\nu}}\, a\right)} e^{i\Omega t}\right)$$

$$\frac{u(r,t)}{U} = \mathrm{Re}\left(\frac{J_1\left(\sqrt{\frac{\Omega a^2}{i\nu}}\left(\frac{r}{a}\right)\right)}{J_1\left(\sqrt{\frac{\Omega a^2}{i\nu}}\right)} e^{i\Omega t}\right)$$

Note: The veloc. profile only depends on the non-dimensional parameter

$$\frac{\Omega a^2}{\nu} \sim \frac{\text{inertial}}{\text{viscosity}}$$

(See animation on webpage)

Consider the case $\Omega \to 0$

Fact:

$$J_1(x) = \frac{1}{2}x - \frac{1}{16}x^3 + \dots$$

as $x \to 0$

$$u(r,t) = \mathrm{Re}\left(U \frac{\cancel{\frac{1}{2}\sqrt{\frac{\Omega}{i\nu}}}\, r + \dots}{\cancel{\frac{1}{2}\sqrt{\frac{\Omega}{i\nu}}}\, a + \dots} e^{i\Omega t} \right)$$

As $\Omega \to 0$

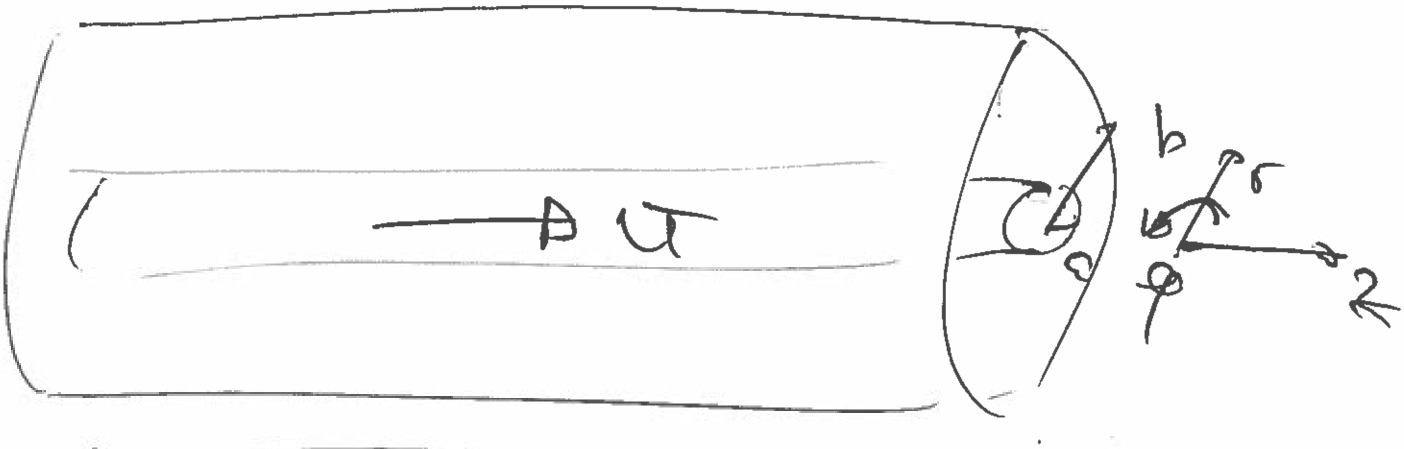
$$u(r,t) \to \frac{r}{a} U \cos(\Omega t)$$

→ rigid body rotation ! [8]

quasi-steady

(as expected).

X-class

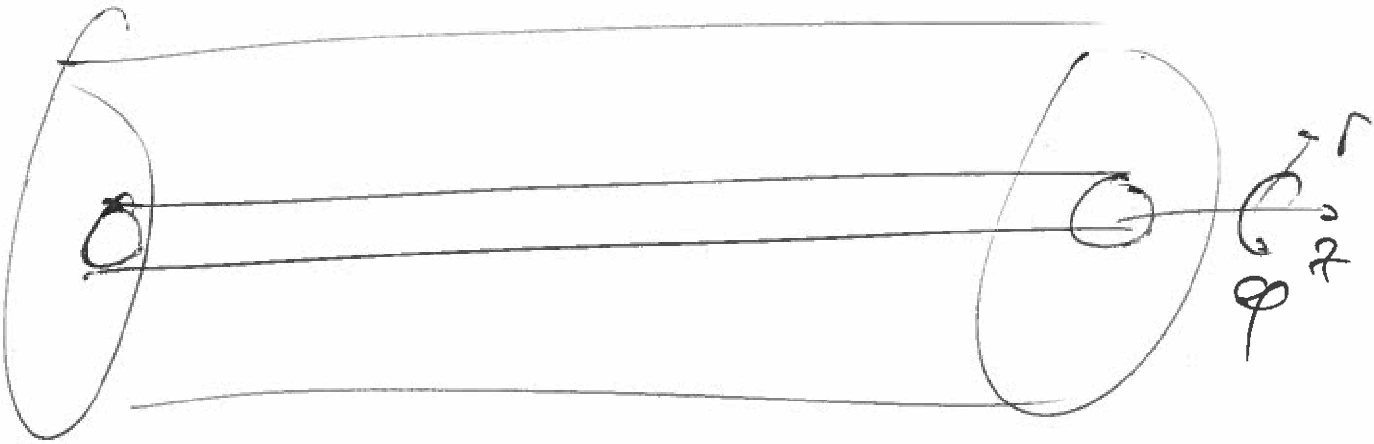


$$\boxed{\nabla p = G\,\underline{e_z}}$$

$$\underline{u}(r, z, \varphi, t) = w(r)\,\underline{e_z}$$

$t_i = \tau_{ij} n_j$

$t_z = ?$



Traction by rod on fluid?

at $r=a$: $\underline{n} = -\underline{e_r}$

$\underline{n} = n_r \underline{e_r} + n_\varphi \underline{e_\varphi} + n_z \underline{e_z}$

$n_r = -1 \qquad n_\varphi = n_z = 0$

$\tau_{ij} = -p\delta_{ij} + 2\mu \underbrace{e_{ij}}$

$\underline{u} = w(r)\, \underline{e_z}$