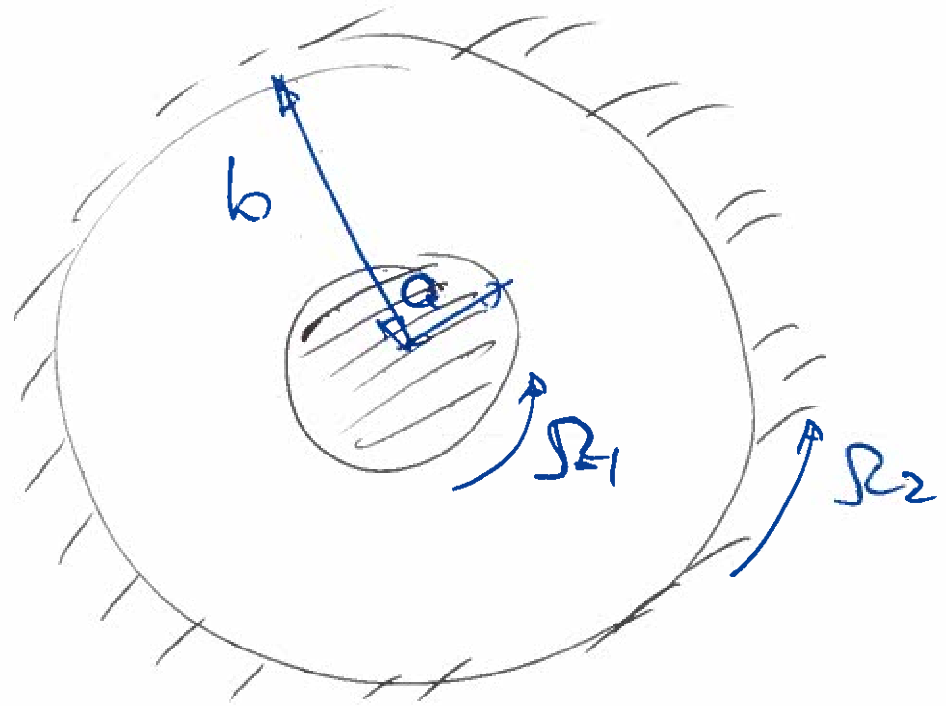
Example: Circular Couette Flow



Assumptions:

- steady
- $\underline{u} = u_\varphi \underline{e_\varphi}$
- $\frac{\partial}{\partial \varphi} = 0$
- $\frac{\partial}{\partial z} = 0$

$$\left.\right\} \quad \underline{u} = u_\varphi(r)\, \underline{e_\varphi}$$

- Flow driven by boundary not a press. gradient

$$\nabla p = \underline{0}$$

(2

$u_r = u = 0$
$u_z = w = 0$
$u_\varphi(r)$

inconsistent!

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \frac{v^2}{r} = \cancel{-\frac{1}{\rho}\frac{\partial P}{\partial r}} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = \cancel{-\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \nu\left[\underbrace{\nabla^2 v - \frac{v}{r^2}} + \cancel{\frac{2}{r^2}\frac{\partial u}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = \cancel{-\frac{1}{\rho}\frac{\partial P}{\partial z}} + \cancel{\nu\nabla^2 w},$$

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

φ - comp. of mom. eqn:

$$0 = \nu \Big(\underbrace{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}}_{\nabla^2 u} - \frac{u}{r^2} \Big)$$

$$u = u_\varphi(r)$$

ODE for $u(r)$:

$$\boxed{r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} - u = 0}$$

Euler ODE.

Solve ~~by~~ ansatz:

$$u \sim r^\lambda$$

into ODE:

$$r^\lambda \underbrace{\big(\lambda(\lambda-1) + \lambda - 1 \big)}_{=0} = 0$$

$$\lambda^2 \cancel{-\lambda} + \cancel{\lambda} - 1 = 0$$

$\lambda = \pm 1$

$\Rightarrow$ gen. form:

$$v(r) = Ar + \frac{B}{r}$$

const. A & B from BCs:

$$v(r=a) = a\Omega_1$$
$$v(r=b) = b\Omega_2$$

$\vdots$

$$v(r) = \frac{1}{b^2-a^2}\left\{\left(b^2\Omega_2 - a^2\Omega_1\right) r - \frac{a^2 b^2 (\Omega_2 - \Omega_1)}{r}\right\}$$

Check: $\Omega_1 = \Omega_2 = \Omega$

$\Rightarrow v(r) = \Omega r$ ✓

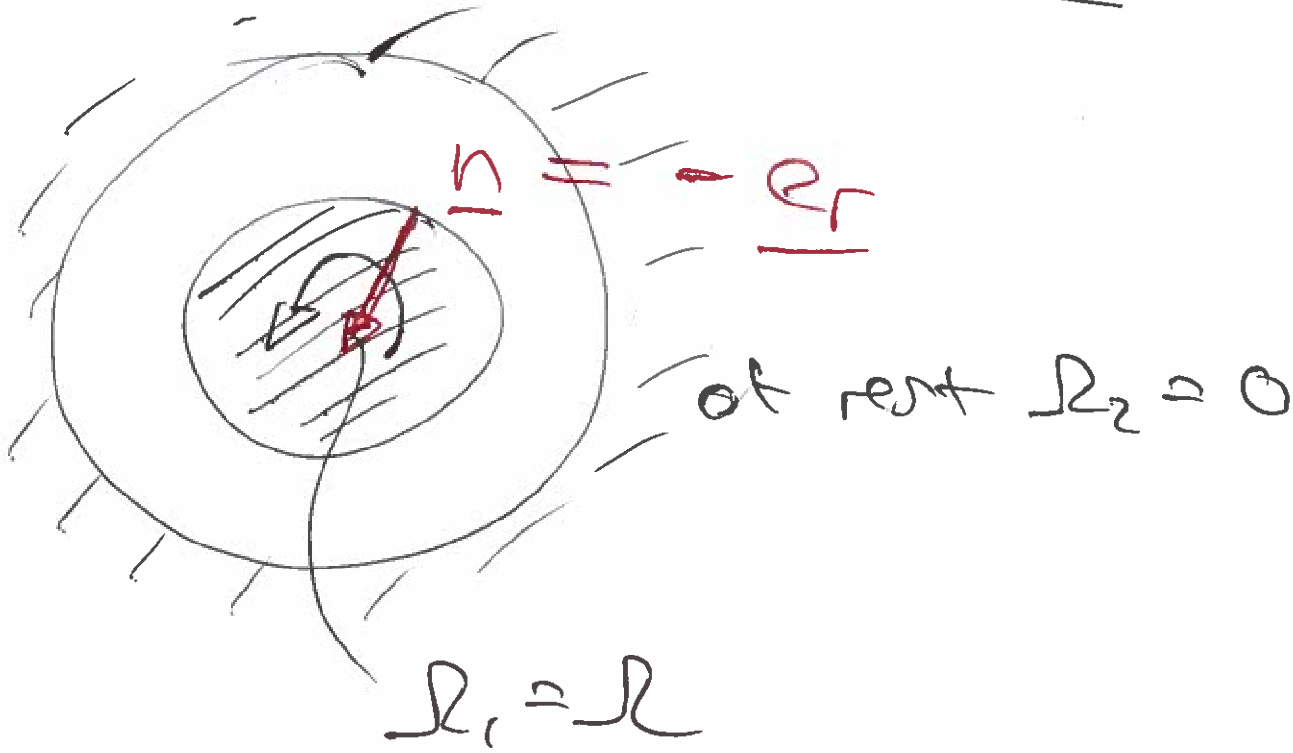
check r-mom. eqn:

$$-\frac{v^2}{r} \overset{!}{=} 0 \qquad \lightning$$

Assumption that $\nabla p = \underline{0}$ was wrong. Rescue by assuming $p = p(r)$

$$\underbrace{\Rightarrow \underbrace{-\frac{v^2}{r}}_{\text{given}} = -\frac{1}{\rho}\frac{\partial p}{\partial r}}_{\text{integrate to get } p(r).}$$

Traction on inner cylinder (or fluid)



$$\boxed{v(r) = \frac{a^2\Omega}{b^2-a^2}\left(\frac{b^2}{r} - r\right)}$$

Traction on fluid at $r = a$:

$$t_i = \underbrace{\tau_{ij}}_{} n_j \qquad i,j = r, \varphi, z$$

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

$$\underline{n} = n_r \underline{e_r} + n_\varphi \underline{e_\varphi} + n_z \underline{e_z} = -\underline{e_r}$$

$$n_r = -1, \quad n_\varphi = n_z = 0$$

7

$u = w = 0$

$v(r)$

$$\cancel{\epsilon_{rr} = \frac{\partial u}{\partial r}} \qquad \epsilon_{\varphi\varphi} = \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{u}{r}}$$

$$\cancel{\epsilon_{zz} = \frac{\partial w}{\partial z}} \qquad \epsilon_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \cancel{\frac{1}{r}\frac{\partial u}{\partial \varphi}}\right]$$

$$\epsilon_{\varphi z} = \frac{1}{2}\left[\cancel{\frac{1}{r}\frac{\partial w}{\partial \varphi}} + \cancel{\frac{\partial v}{\partial z}}\right] \qquad \epsilon_{rz} = \frac{1}{2}\left[\cancel{\frac{\partial u}{\partial z}} + \cancel{\frac{\partial w}{\partial r}}\right]$$

$$\varepsilon_{r\varphi} = \frac{1}{2} r \frac{\partial}{\partial r}\left(\frac{v}{r}\right)$$

$\vdots$

$$\varepsilon_{r\varphi} = -\frac{a^2 b^2 \Omega}{b^2 - a^2}\frac{1}{r^2}$$

At $r = a$: $\boxed{\varepsilon_{r\varphi}\big|_{r=a} = -\frac{b^2\Omega}{b^2 - a^2}}$

$$t_i = -p n_i + 2\mu \varepsilon_{ij} n_j$$

$i = "r"$:

$$t_r = -p n_r + 2\mu \left(\cancel{\varepsilon_{rr} n_r} + \varepsilon_{r\varphi} \cancel{n_\varphi} + \varepsilon_{rz} \cancel{n_z} \right)$$

$$n_r = -1 \quad n_\varphi = n_z = 0$$

$$\boxed{t_r = p(r=a)}$$

$i = "\varphi"$

$$t_\varphi = \cancel{-p n_\varphi} + 2\mu \left(\varepsilon_{\varphi r} n_r + \cancel{\varepsilon_{\varphi\varphi} n_\varphi} + \cancel{\varepsilon_{\varphi z}} n_z \right)$$

$$\boxed{t_\varphi = -2\mu \varepsilon_{\varphi r} \Big|_{r=a} = 2\mu \frac{b^2 \Omega}{b^2 - a^2}}$$

Sheer stress acting between fluid & solid of inner cylinder.