$$\boxed{\begin{aligned} \rho \frac{D\underline{u}}{Dt} &= -\nabla p + \mu \nabla^2 \underline{u} \\ \nabla \cdot \underline{u} &= 0 \end{aligned}}$$

$$\underline{u} = u_1 \underline{e_1} + u_2 \underline{e_2} + u_3 \underline{e_3}$$

$$\nabla = \dots$$

or

$$\underline{u} = u_r \underline{e_r} + u_\varphi \underline{e_\varphi} + u_z \underline{e_z}$$

$$\nabla = \dots$$

Then extract r, φ, z comp. from momentum eqns.

$\Rightarrow$ A mess! See handout!

Can still use index notation

$$t_i = \tau_{ij} n_j$$

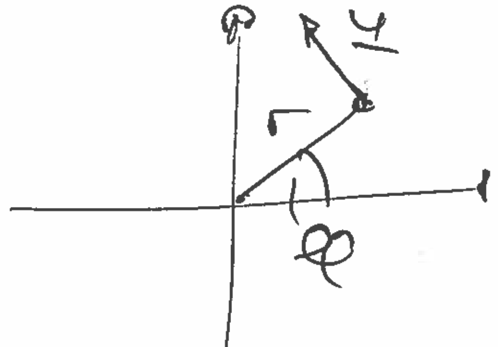
$$i, j = "r", "\varphi", "z"$$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$$

where do these "extra" terms come from?

Illustrate: for rigid body rotation:

$$\underline{u} = \Omega r \, \underline{e}_\varphi$$



$$\underline{u} = \underbrace{u_r}_{=0} \underline{e}_r + \underbrace{u_\varphi}_{=\Omega r} \underline{e}_\varphi + \underbrace{u_z}_{=0} \underline{e}_z$$

$$u_r = u \qquad u_\varphi = v \qquad u_z = w$$

into r-component of N.St:

$$\rho \frac{v^2}{r} = \frac{\partial p}{\partial r}$$

$$\rho \frac{\Omega^2 r^2}{r} = \frac{\partial p}{\partial r}$$

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$$\underline{u} = u\,\underline{e_r} + v\,\underline{e_\varphi} + w\,\underline{e_z}$$

$u=0 \qquad v=\Omega r \qquad w=0$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} \boxed{- \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right],$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0.$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

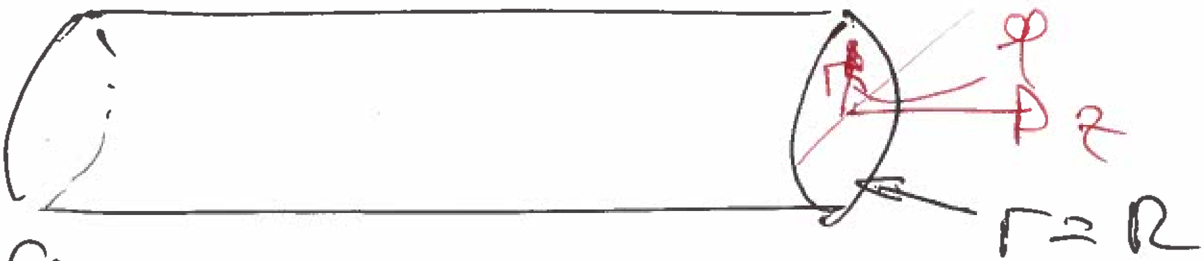
$$\frac{dp}{dr} = \rho \Omega^2 r$$

$$p = p_0 + \frac{1}{2}\rho \Omega^2 r^2$$

This term represents centrifugal effects.

Example:

Hagen-Poiseuille flow in a pipe



flow driven by press. gradient.

Assume: Flow is indep of z, indep of φ, unidirectional & steady.

$$\underline{u} = u_z \underline{e_z} = u_z(r)\, \underline{e_z}$$

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$p = p(\cancel{r}, z)$ $\quad w(r)$

z-component:

$$0 = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{fct of } z} + \underbrace{\nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)}_{\text{fct of } r}$$

clash!

$\Longrightarrow$ $\frac{\partial p}{\partial z}$ is indep of z

$$\frac{\partial p}{\partial z} = G = \text{const.}$$

$$\frac{G}{\mu} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$$

integrate:

$$\frac{1}{2}\frac{G}{\mu}r^2 + A = r\frac{dw}{dr}$$

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$u = v = 0$

$w = w(r) \qquad P = P(\cancel{r}, z)$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \cancel{\frac{v^2}{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \cancel{\nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right]},$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = \cancel{-\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \cancel{\nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right]},$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

where

$$\nabla^2 \equiv \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}} + \cancel{\frac{\partial^2}{\partial z^2}}.$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}$$

again

$$\frac{1}{2}\frac{G}{\mu}r + \frac{A}{r} = \frac{dw}{dr}$$

$$\frac{1}{4}\frac{G}{\mu}r^2 + A\ln r + B = w(r)$$

2 const. A, B from BCs:

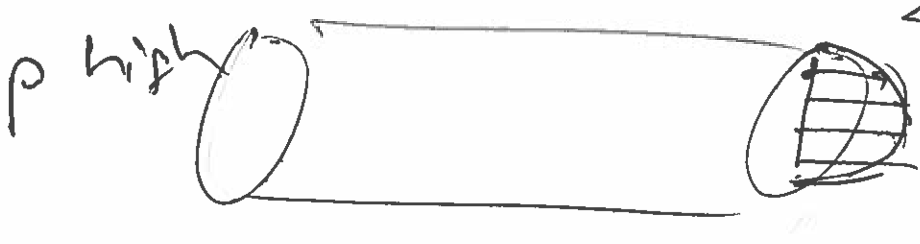
$\underline{u}(r=R) = \underline{0}$

$w(r=R) = 0$

Also veloc must stay finite

$\Rightarrow A = 0.$

$$w(r) = \frac{1}{4}\frac{G}{\mu}\underbrace{(r^2 - R^2)}_{<0}$$



$w > 0 \Rightarrow G < 0$

$$G = \frac{\partial p}{\partial z} < 0$$

Ex-CLASS

$u(y=0) = -U$

$y=h$: $\boxed{t_i = \tau_{ij} n_j}$

applied traction outer unit normal

$n_1 = 0$

$n_2 = 1$

$$\underline{t} = \underbrace{-\tau_0 \underline{e_x}} = t_1 \underline{e_x} + t_2 \underline{e_y}$$

$$t_1 = -\tau_0 \qquad t_2 = 0$$

$$t_i = -p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

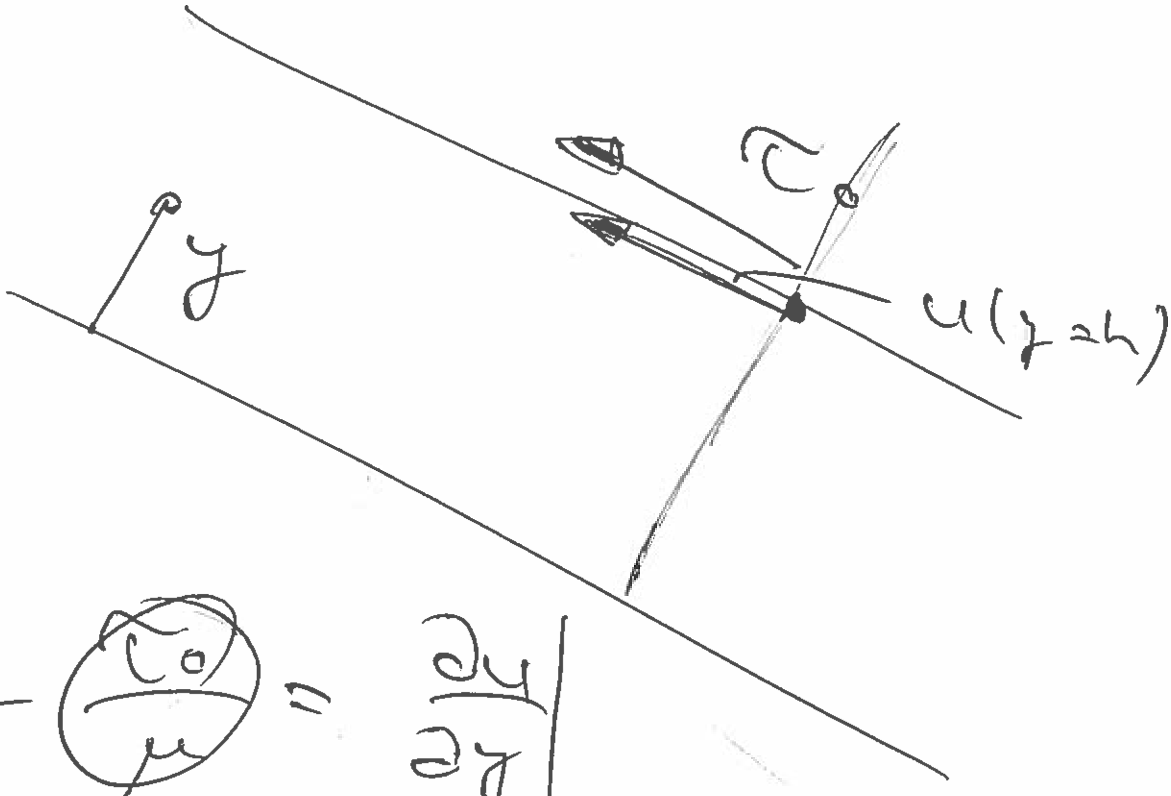
$i=1$

$$t_1 = -p \cancel{n_1} + \cancel{\mu\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) n_1}$$

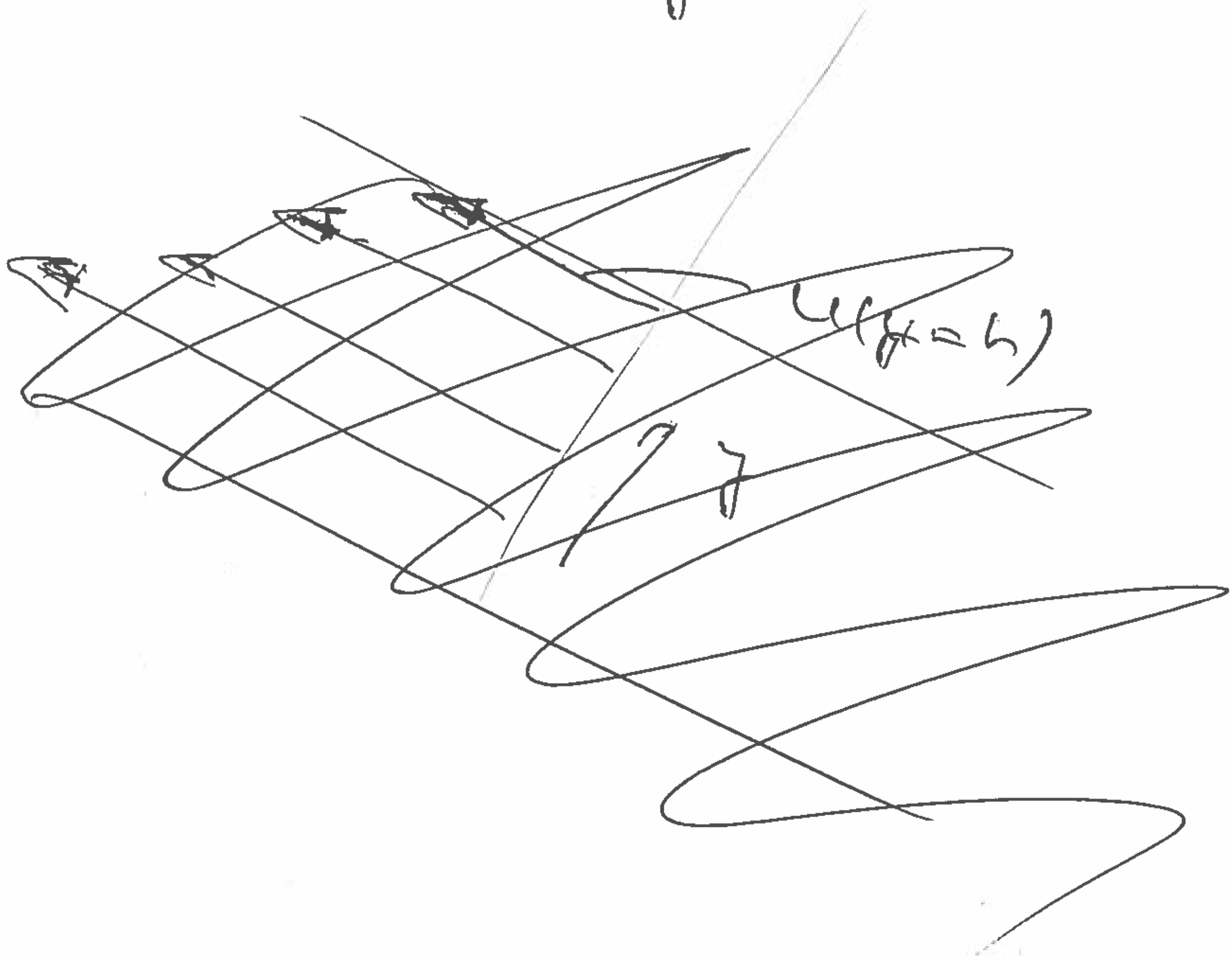
$$+ \mu\left(\frac{\partial u_1}{\partial x_2} + \cancel{\frac{\partial u_2}{\partial x_1}}\right) n_2 \qquad n_2 = 1$$

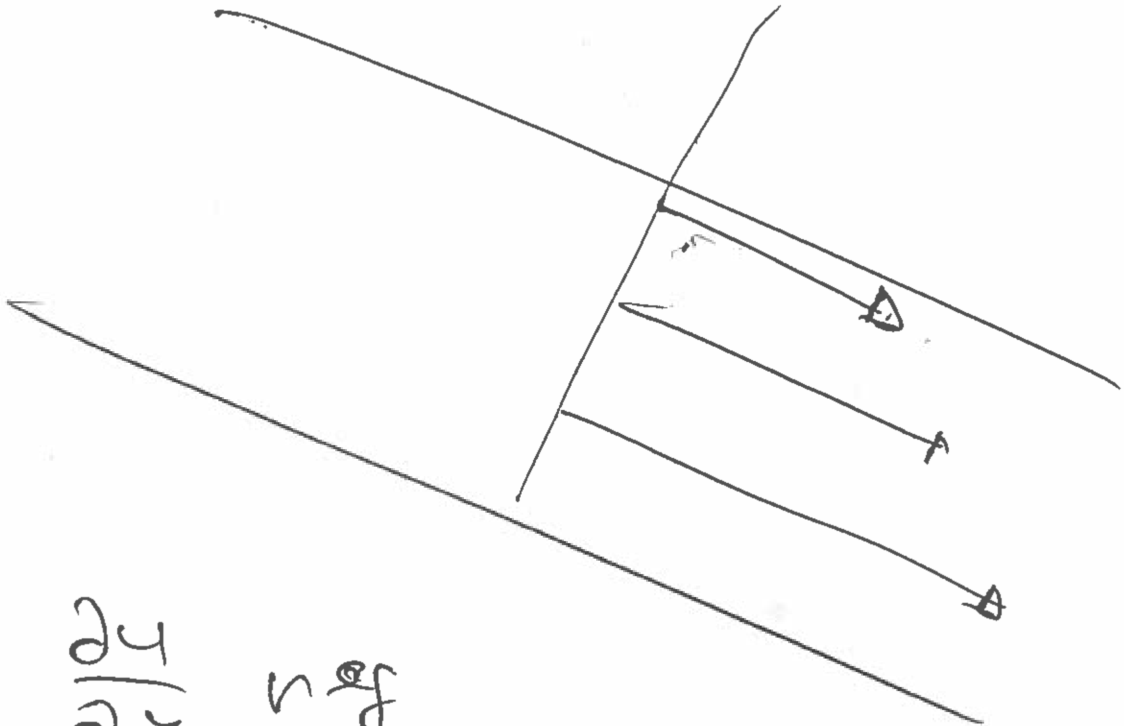
$$t_1 = \mu \frac{\partial u_1}{\partial x_2} = -\tau_0$$

$$\boxed{-\tau_0 = \mu \frac{\partial u}{\partial y}} \quad \text{at } y=h$$



$$\underbrace{-\frac{\tau_0}{\mu}}_{\text{neg.}} = \underbrace{\left.\frac{\partial u}{\partial y}\right|_{y=h}}_{\text{neg.}}$$





$\frac{\partial u}{\partial y}$ ng