$$\underline{u} = u(\cancel{x}, y, z, t)\, \underline{e_x}$$

$$\rho \frac{\partial u}{\partial t} = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho f_y = \frac{\partial p}{\partial y}$$

$$\rho f_z = \frac{\partial p}{\partial z}$$

$\underline{f} = \underline{0}$:

$$\frac{\partial u}{\partial t} = -\frac{G(t)}{\rho} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial p}{\partial x} = G(t)$$

Example: The vibrating plate

Assume:

$$u(y,z,t) = u(y,t)$$

no need for press. gradient: $G=0$

$$\frac{\partial u}{\partial t} = -\cancel{\frac{G(t)}{\rho}} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}} \qquad u(y,t)$$

BC: $u = U\cos(\omega t)$ at $y=0$
(no slip)

$u \to 0$ as $y \to \infty$.

IC: $u(y, t=0) = ?$

Alternative: Assume flow is time-periodic

$$u(y,t) = f(y)\cos(\omega t + \phi(y))$$

Easier to use complex variables

$$u(y,t) = \mathrm{Re}\left(f(y)\, e^{i\omega t}\right)$$

insert into PDE:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$i\omega f = \nu f''$$

$$\boxed{f'' - \frac{i\omega}{\nu} f = 0} \qquad f(y)$$

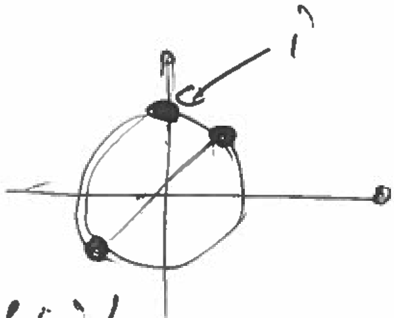
const. coeffn. ODE for $f(y)$:

$$f \sim e^{\lambda y}$$

char. poly:

$$\lambda^2 - \frac{i\omega}{\nu} = 0$$

$$\lambda = \pm\sqrt{\frac{\omega}{\nu}}\,\underbrace{\sqrt{i}}_{\pm\frac{1}{2}\sqrt{2}(1+i)}$$



$$\lambda = \pm(1+i)\sqrt{\frac{\omega}{2\nu}}$$

$$f(y) = A \exp\left((1+i)\sqrt{\frac{\omega}{2\nu}}\, y\right) + B \exp\left(-(1+i)\sqrt{\frac{\omega}{2\nu}}\, y\right)$$

2BC for A, B:

$$u(y=0) = U\cos(\omega t) = \mathrm{Re}\left(U e^{i\omega t}\right)$$

$$\Rightarrow f(y=0) = U$$

$$\boxed{U = A + B}$$

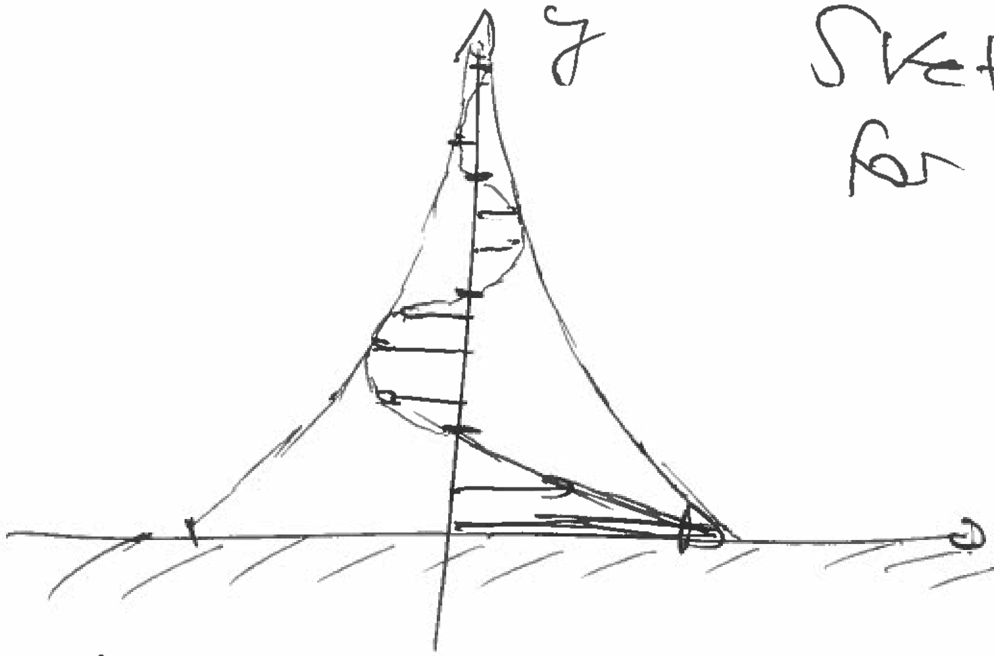
$u \to 0$ as $y \to \infty$: $f \to 0$ as $y \to \infty$

$$\Rightarrow \boxed{A = 0} \; ; \; \boxed{B = U}$$

$$u(y,t) = \mathrm{Re}\left(U e^{-(1+i)\sqrt{\frac{\omega}{2\nu}}\, y}\, e^{i\omega t}\right)$$

$$\boxed{u(y,t) = U e^{-\sqrt{\frac{\omega}{2\nu}}\, y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\, y\right)}$$

EXERCISE



Sketch for fixed t

Note: Decay of the exponential is controlled by $\sqrt{\frac{\omega}{2\nu}} = \delta$.

This number also controls the width of the layers of fluid above the plate.

$$\delta = \sqrt{\frac{\rho\omega}{2\mu}} = \sqrt{\frac{\text{inertia}}{\text{viscosity}}}$$

See animation on webpage.

§(n+1) ~~curvilien~~ curvilinear coordinates

So far:

Cartesian coordinates:
for coordinates & bases of any vectors.

$\underline{u} = u_1 \underline{e_1} + u_2 \underline{e_2} + u_3 \underline{e_3}$

$\Rightarrow u_i$

Transform. eqns to a different coord. system

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$x = r\cos\varphi$

$y = r\sin\varphi$ ...

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2}$$

This is for a scalar.

N. St. eqns. contain differential operators acting on vector. In that case we must differentiate the basis vectors too!

$$\underline{u} = u_1 \underline{e_1} + u_2 \underline{e_2} + u_3 \underline{e_3}$$

$$= u_r \underline{e_r} + u_\varphi \underline{e_\varphi} + u_z \underline{e_z}$$

the basis vectors depend on $(\cancel{r}, \varphi, \cancel{z})$

$\Longrightarrow$ A MESS ☹