$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

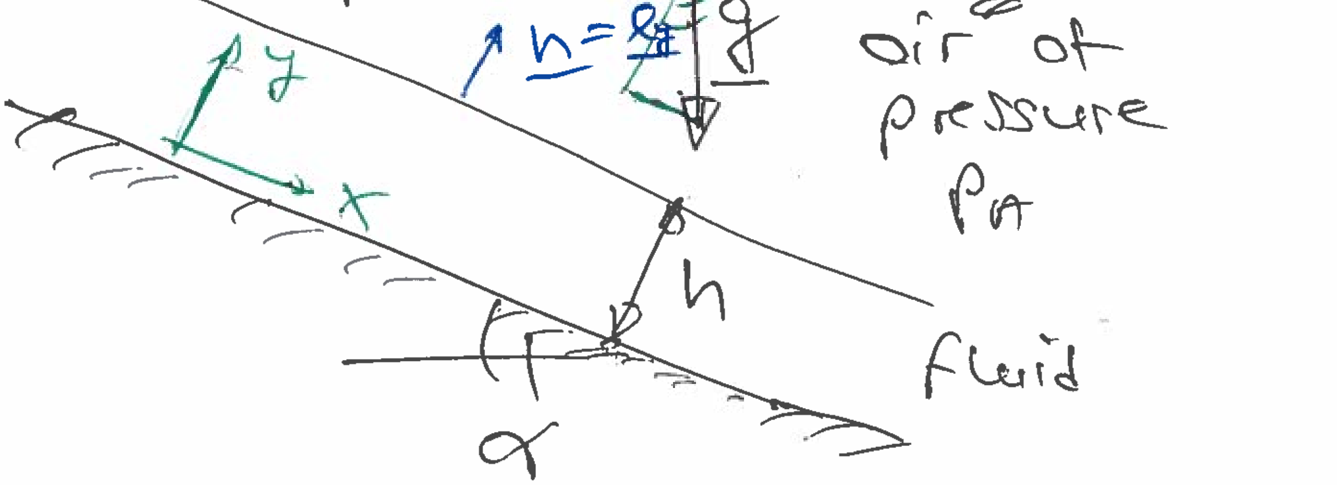
& $$\frac{\partial u_k}{\partial x_k} = 0 \qquad + BC/IC$$

Much too hard !

$$\underline{u} = u(\cancel{x}, y, z, t)\,\underline{e_x}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} &= \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ \frac{\partial p}{\partial y} &= \rho f_y \qquad \frac{\partial p}{\partial z} = \rho f_z \end{aligned}}$$

Example: Flow down an inclined plane



Decompose $\underline{g}$ into x & y direction

$$\underline{F} = \underline{g} = \overbrace{g\sin\alpha}^{F_x}\,\underline{e_x} - \overbrace{g\cos\alpha}^{F_y}\,\underline{e_y}$$

Assume: $u(\cancel{x},y,\cancel{z},\cancel{t}) = u(y)$ (2

Into parallel flow eqn:

$$\cancel{\frac{\partial u}{\partial t}} = -\frac{\partial p}{\partial x} + \underbrace{\rho g \sin\alpha}_{f_x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha} \quad (1)$$

y-comp:

$$0 = -\frac{\partial p}{\partial y} \underbrace{- \rho g \cos\alpha}_{f_y} = 0$$

$$\boxed{\frac{\partial p}{\partial y} = -\rho g \cos\alpha} \quad (2)$$

z-comp:

$$\boxed{\frac{\partial p}{\partial z} = \underset{f_z}{0}} \Longrightarrow \boxed{p = p(x,y)}$$

BC: $y=0$: no slip:

$$u(y=0)=0$$

$y=h$: traction BC:

Traction applied by inviscid air onto the fluid is:

$$\underline{t} = -p_A \underline{n} = -p_A \underline{e_y}$$

Traction BC:

$$\tau_{ij} n_j = t_i$$

$\underline{n} = \underline{e_y}$:

$$n_1 = n_x = 0$$
$$n_2 = n_y = 1$$
$$n_3 = n_z = 0$$

$\underline{t} = -p_A \underline{e_y}$

$$t_1 = t_x = 0$$
$$t_2 = t_y = -p_A$$
$$t_3 = t_z = 0$$

$$\tau_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

write down

$\tau_{ij} n_j = t_i$ for $i = 1, 2, 3$:

$$t_i = -p n_i + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j \quad \text{at } y = h$$

i = 2:

$$t_2 = -p_A = -p \overset{1}{n_2} + \mu \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \cancel{n_1}^{0} +$$

$$\cancel{+ \mu \left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right)} \overset{1}{n_2} +$$

because $u_2 = 0$

$$+ \mu \left(\qquad \right) \underset{0}{\cancel{n_3}}$$

$-p_A = -p$ at $y = h$

$$\boxed{p(y=h) = p_A}$$

5

$i=1$:

$$t_1 = 0 = -p\cancel{n_1}^{0} + \mu(\ldots)\cancel{n_1}^{0} +$$

$$\mu\left(\frac{\partial u_1}{\partial x_2} + \cancel{\frac{\partial u_2}{\partial x_1}}\right)n_2 +$$

$$+\mu(\ldots)\cancel{n_3}^{0}$$

$u_2=0$

$$\boxed{0 = \mu\frac{\partial u}{\partial y}} \quad \text{at } y=h$$

$i=3$:

$$0=0 \quad \checkmark \text{ (check)}$$

Solve (2): integrate w.r.t y

$$\frac{\partial p}{\partial y} = -\rho g\cos\alpha$$

$$p = -\rho g y\cos\alpha + f(x)$$

Apply BC:

$$p(y=h) = p_A \overset{!}{=} -\rho g h \cos\alpha + f(x)$$

$$\Rightarrow \boxed{p(x,y) = p_A + \rho g \cos\alpha \, (h-y)}$$

- hydrostatic press distribution; pressure increases with depth below free surface.
- press. does not depend on x; flow not driven by pressure!

Now (1):

$$0 = -\cancel{\frac{\partial p}{\partial x}} + \underbrace{\rho g \sin\alpha}_{\text{const.}} + \mu \frac{\partial^2 u}{\partial y^2}$$

Integrate twice w.r.t. y

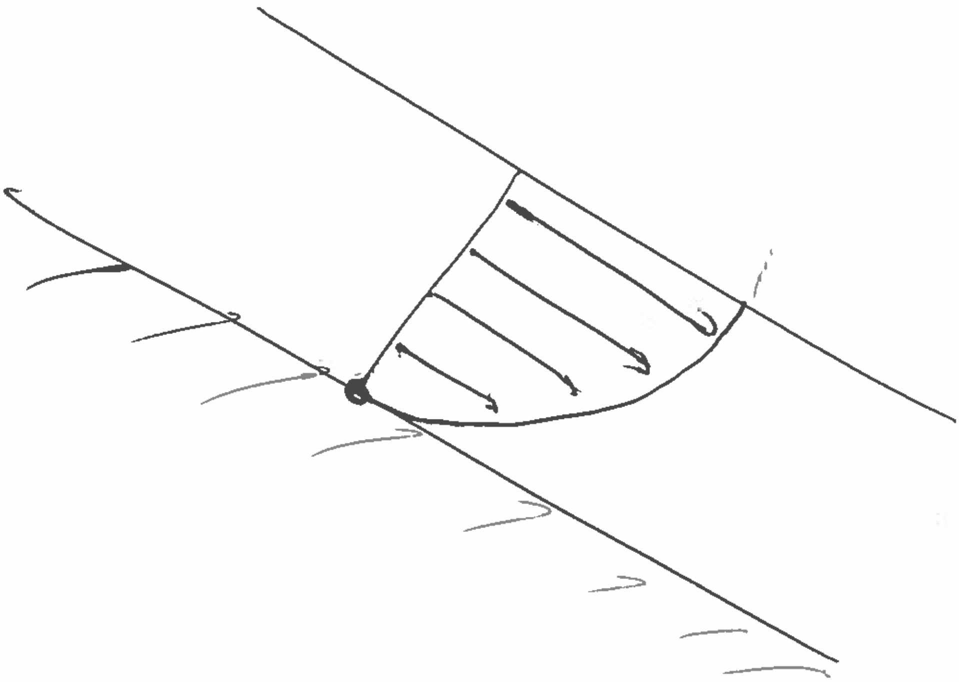
$$u = -\frac{1}{2}\frac{\rho g \sin\alpha}{\mu} y^2 + Ay + B$$

Apply BC:

$$u(y=0)=0$$

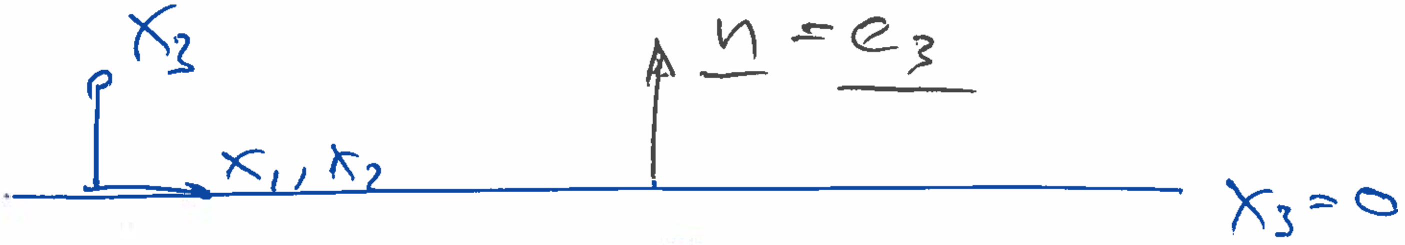
$$\mu \left.\frac{\partial u}{\partial y}\right|_{y=h}=0$$

$$\Rightarrow \boxed{u=\frac{\rho g \sin\theta}{\mu}\left(hy-\frac{1}{2}y^2\right)}$$



X-cuts:

$n_1 = 0, \quad n_2 = 0, \quad n_3 = 1$



Newtonian fluid 1

μ, ρ

$$\tau_{ij} n_j = t_i$$

Stress tensor in medium 1

outer unit normal on medium 1

traction applied onto medium 1

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$t_i = -p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

sum over j
but only $n_3 = 1$
$\Rightarrow$ only need $j = 3$

$i=1$

$$t_1 = -p\underbrace{n_1}_{0} + \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\underbrace{n_3}_{1}$$

$$\boxed{t_1 = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)}$$

$i=2$:

$$t_2 = -p\underbrace{n_2}_{0} + \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)\underbrace{n_3}_{1}$$

$$\boxed{t_2 = \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)}$$

$i=3$:

$$t_3 = -p\underbrace{n_3}_{1} + \mu\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\right)\underbrace{n_3}_{1}$$

$$\boxed{t_3 = -p + 2\mu\frac{\partial u_3}{\partial x_3}}$$

$$\underline{t} = -P\,\underline{n} = -P\,\underline{e}_3$$

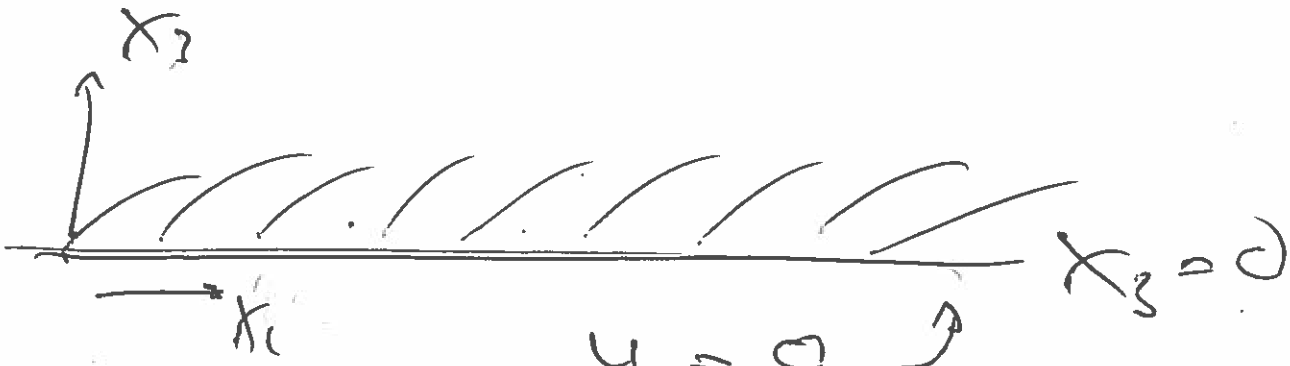
$$t_1 = 0$$
$$t_2 = 0$$
$$t_3 = -P$$

$$t_1 = 0 = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$t_2 = 0 = \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$t_3 = -P = -p + 2\mu\frac{\partial u_3}{\partial x_3} \quad \text{at } x_3 = 0$$

$$\boxed{p(x_3 = 0) = P + \underbrace{2\mu\frac{\partial u_3}{\partial x_3}}_{\text{viscous normal stresses}}}$$



$\underline{u} = \underline{0}$

$u_1 = u_2 = u_3$ at $x_3 = 0$

$\forall x_1, x_2$

$\frac{\partial u_i}{\partial x_{1,2}} = 0$

$$t_1 = \mu\left(\frac{\partial u_1}{\partial x_3} + \cancel{\frac{\partial u_3}{\partial x_1}}\right)$$

$$t_1 = \mu \frac{\partial u_1}{\partial x_3}$$

$$t_2 = \mu\left(\frac{\partial u_2}{\partial x_3} + \cancel{\frac{\partial u_3}{\partial x_2}}\right)$$

$$t_2 = \mu \frac{\partial u_2}{\partial x_3}$$

$$t_3 = -p + 2\mu \cancel{\frac{\partial u_3}{\partial x_3}}$$

$$\frac{\partial u_k}{\partial x_k} = \cancel{\frac{\partial u_1}{\partial x_1}} + \cancel{\frac{\partial u_2}{\partial x_2}} + \boxed{\frac{\partial u_3}{\partial x_3} = 0}$$