Chapter 1

Introduction

This set of notes summarises the main results of the lecture 'Viscous Fluid Flow' (MATH35001). Please email any corrections (yes, there might be the odd typo...) or suggestions for improvement to

M.Heil@maths.manchester.ac.uk

Alternatively, see me after the lecture or in my office (Room 2.224 in the Alan Turing building).

Generally, the notes will be handed out after the material has been covered in the lecture. You can also download them from the WWW:

http://www.maths.manchester.ac.uk/˜mheil/Lectures/Fluids/.

This WWW page will also contain announcements, example sheets, solutions, etc.

1.1 Literature

The following is a list of books that I found useful in preparing this lecture. **It is not necessary to purchase any of these books!** Your lecture notes and these handouts will be completely sufficient.

Acheson, D.J. 1990 Elementary Fluid Dynamics. Clarendon Press, Oxford, 1990.

Spiegel, M. 1974 Vector Calculus. McGraw Hill (Schaum's Outline series).

Batchelor, G.K. 1967 An Introduction to Fluid Dynamics. Cambridge.

Sherman, F.S. 1990 Viscous Flow. McGraw Hill.

McCormack , P.S. & Crane, L.J. 1973 Physical Fluid Dynamics, Academic Press.

Panton, R.L. 1996 Incompressible Flow (second edition), Wiley.

White, F.M. 1991 Viscous Fluid Flow (second edition), McGraw Hill.

1.2 Preliminaries: Index notation & summation convention

- We will denote vectors/matrices/tensors by their components relative to a set of basis vectors. E.g. instead of writing $\mathbf{r} = r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 = (r_1, r_2, r_3)$, we simply write r_i and use the convention that all 'free indices' (i in this case) range from 1 to 3. Similarly, we represent the matrix
$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
by its generic component a_{ij} and imply that the two free indices (i and j) take on all values in the range from 1 to 3.

- In general, we only write down the generic term of any vector (or vector equation). For instance $a_i = b_i + c_i$ is taken to represent the three components $a_1 = b_1 + c_1, a_2 = b_2 + c_2, a_3 = b_3 + c_3$ of the symbolic vector equation $\mathbf{a} = \mathbf{b} + \mathbf{c}$.
- Consistency check: Every term in an equation in index notation has to have the same number of 'free indices'. For instance, the addition of two matrices can be expressed as $A_{ij} = B_{ij} + C_{ij}$, whereas the equation $A_{ij} = B_{ik} + C_{lm}$ does not make sense.
- Kronecker Delta: $\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$
- Summation convention: Automatic summation over repated indices. Examples are:

  Dot product: $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_i$. Sums like this will occur very frequently and it will turn out to be convenient to drop the summation sign and to automatically sum over any repeated index. I.e. $\sum_{i=1}^{3} a_i b_i = a_i b_i = a_k b_k$. Note that the 'name' of the summation index is irrelevant as it does not appear in the final result; therefore $a_i b_i$ is the same as $a_k b_k$. Summation indices are often called 'dummy indices'.

  Matrix-vector products: $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ becomes $A_{ij} x_j$ (or $A_{im} x_m = b_i$, say). Similarly $\mathbf{A}^T \cdot \mathbf{x} = \mathbf{c}$ becomes $A_{ji} x_j = c_i$ (or $A_{jk} x_j = c_k$, say). Note that the result of the matrix-vector product is a vector: Hence both sides of the equations have one (matching!) free index.

  δ_{ij} **'exchanges' indices:** $a_i \delta_{ij} = a_j$.
- Comma denotes partial differentiation: E.g. $\frac{\partial u_i}{\partial x_j} = u_{i,j}$.
- Some differential operators in index notation:

$$\nabla \cdot \mathbf{u} = \text{div } \mathbf{u} = u_{i,i} \tag{1.1}$$

$$\nabla \phi = \text{grad} \phi = \phi_{,i} \tag{1.2}$$

$$\nabla^2 \phi = \phi_{,ii} \tag{1.3}$$