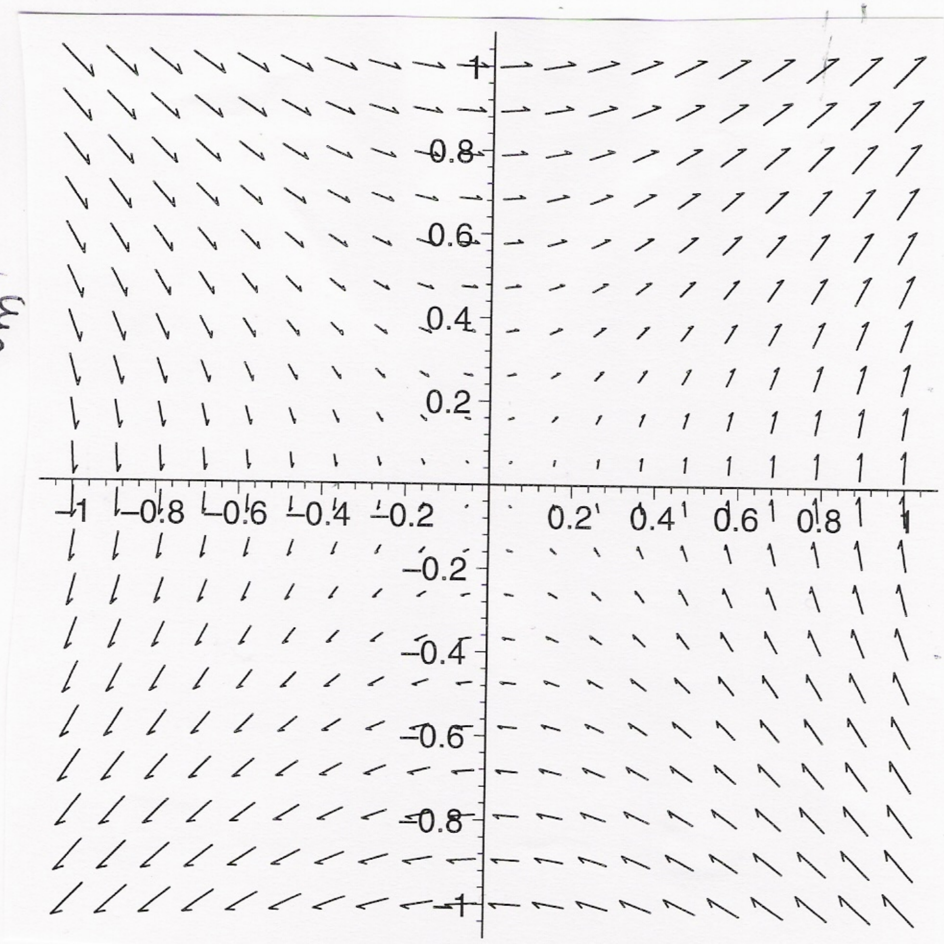
EXAMPLE SHEET II

1) a)



Second diagonal:

$x_1 = -x_2 = \xi$

$\underline{u} = a\xi \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{x_2 = 0}$:

$u_1 = 0$

$u_2 = a x_1$

main diagonal:

$x_1 = x_2 = \xi$

$\underline{u} = a\xi \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{x_1 = 0}$: $u_1 = a x_2$, $u_2 = 0$

b) Eqn. of motion:

$$u_1 = a x_2^p = \frac{\partial x_1^p}{\partial t} \qquad (1)$$

$$u_2 = a x_1^p = \frac{\partial x_2^p}{\partial t} \qquad (2)$$

$\frac{\partial}{\partial t}(1)$ into (2)

$$a x_1^p = \frac{\partial x_2^p}{\partial t} = \frac{1}{a}\frac{\partial^2 x_1^p}{\partial t^2}$$

$$\ddot{x}_1^p - a^2 x_1^p = 0$$

$$\boxed{x_1^p(t) = Ae^{at} + Be^{-at}}$$

Also from (1):

$$\boxed{x_2^p = \frac{1}{a}\frac{\partial x_1^p}{\partial t} = Ae^{at} - Be^{-at}}$$

initial cond:

At time $t=0$: $\quad x_i^p = X_i$

$\Rightarrow \quad \left.\begin{array}{l} A+B = X_1 \\ A-B = X_2 \end{array}\right\} \quad \begin{array}{l} A = \frac{1}{2}(X_1+X_2) \\ B = \frac{1}{2}(X_1 - X_2) \end{array}$

$$\boxed{\begin{array}{l} x_1^p(t) = \frac{1}{2}(X_1+X_2)e^{at} + \frac{1}{2}(X_1-X_2)e^{-at} \\ x_2^p(t) = \frac{1}{2}(X_1+X_2)e^{at} - \frac{1}{2}(X_1-X_2)e^{-at} \end{array}}$$

This gives the spatial posn. of a particle in the flow at time t which was at X_i at time $t=0$.

c) The acceleration of the individual particle is

$$a_i = \frac{\partial^2 x_i^p}{\partial t^2}$$

w.l.o.g. we can consider the situation at $t=0$ (the flow is steady!).

$\exists$ The particle is at $x_i = X_i$

& we have

$$a_1 = \frac{\partial^2 x_1^p}{\partial t^2}(x_i = X_i) = a^2 X_1 = a^2 x_1$$

$$a_2 = \frac{\partial^2 x_2^p}{\partial t^2}(x_i = X_i) = a^2 X_2 = a^2 x_2$$

d) using the substantial derivative, we obtain:

$$a_i = \cancel{\frac{\partial u_i}{\partial t}} + u_j \frac{\partial u_i}{\partial x_j}$$

steady flow

$$a_1 = u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} =$$

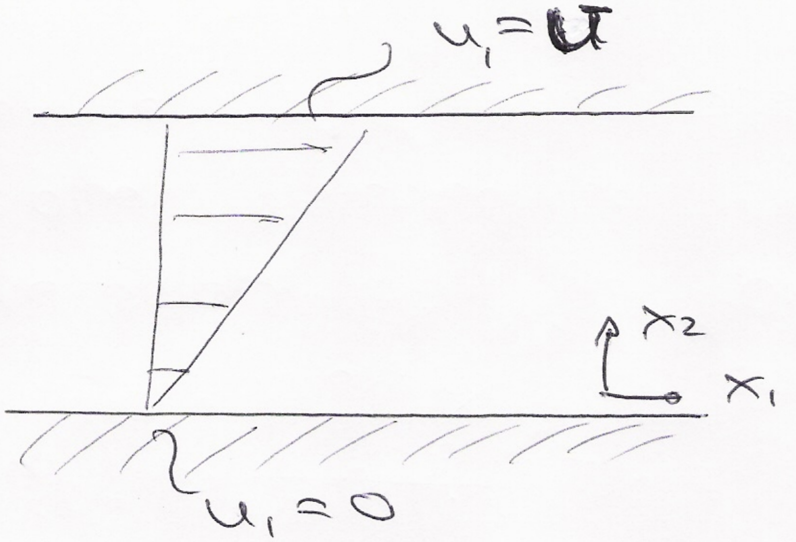
$$= a x_2 \cdot 0 + a x_1 a = a^2 x_1 \quad \checkmark$$

$$a_2 = u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} =$$

$$= a x_2 a + a x_1 \cdot 0 = a^2 x_2 \quad \checkmark$$

Remember: in both cases, we are interested in the acceleration of the particle that is <u>currently</u> at a given spatial position. This is <u>not</u> the same as $\frac{\partial u}{\partial t}$ (which is zero in the present case!)

2) a)



$u_1 = U x_2$

$u_2 = 0$

This flow field could be the flow in a 2D channel with a stationary bottom wall (at $x_2 = 0$) & a moving top wall which moves to the right with velocity U.

b)

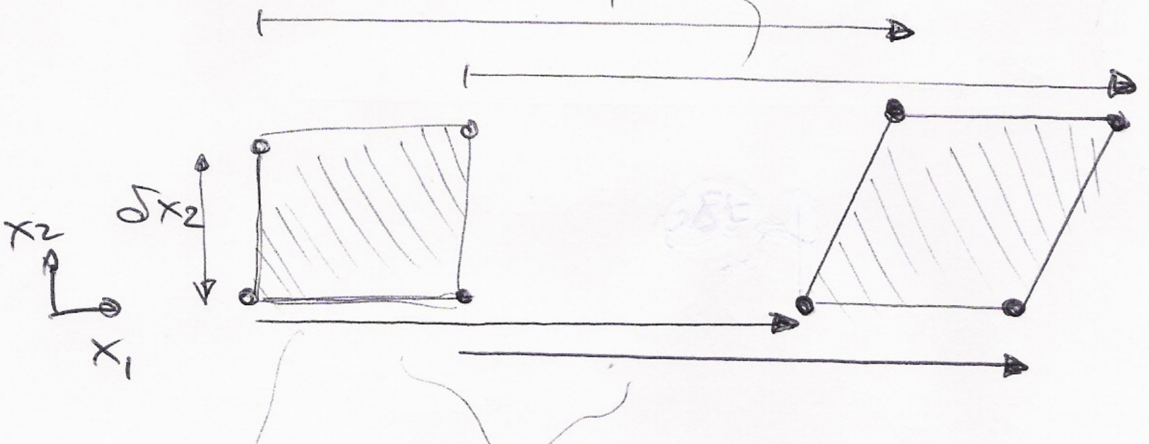
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\frac{\partial u_i}{\partial x_j} = \begin{array}{c|c} & \begin{array}{cc} j=1 & j=2 \end{array} \\ \hline \begin{array}{c} i=1 \\ i=2 \end{array} & \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} \end{array}$$

$$\varepsilon_{ij} = \begin{pmatrix} 0 & \frac{1}{2}U \\ \frac{1}{2}U & 0 \end{pmatrix} \qquad \omega_{ij} = \begin{pmatrix} 0 & \frac{1}{2}U \\ -\frac{1}{2}U & 0 \end{pmatrix}$$

$$u_1(X_2+\delta x_2)\,\delta t = U(X_2+\delta x_2)\,\delta t$$

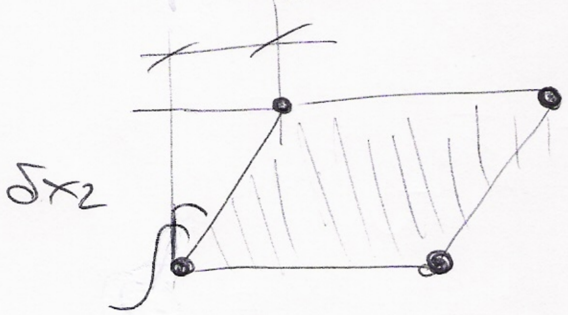


NO AXIAL EXTENSION ($\varepsilon_{11} = 0$)

$$u_1(x_2 = X_2)\,\delta t = U X_2$$

(lower edge at $x_2 = X_2$, say)

$U\,\delta x_2\,\delta t$



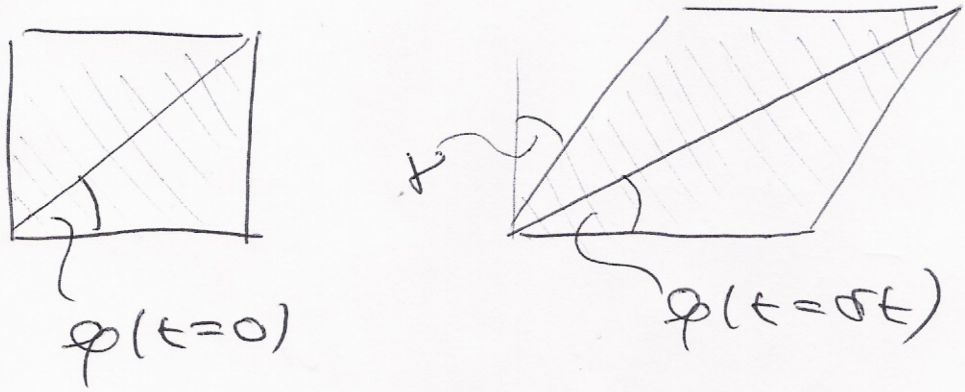
$$\tan\gamma \approx \gamma = \frac{U\,\cancel{\delta x_2}\,\delta t}{\cancel{\delta x_2}}$$

$$\frac{D\gamma}{Dt} = U = 2\,\varepsilon_{12}$$

✓, just as claimed ☺

Also consider the rotation of a (square) element by following its main diagonal:

$t=0$ $\qquad\qquad$ $t=\delta t$



($\varphi = \frac{\pi}{4}$ for a square elem.)

$$\varphi = \frac{1}{2}\left(\frac{\pi}{2} - \gamma\right)$$

$$\boxed{\frac{D\varphi}{Dt} = -\frac{1}{2}\dot{\gamma}}$$

The main diagonal rotates about the x_3 axis (which comes out of the paper!) with a (neg.) angular velocity

$$\frac{D\varphi}{Dt} = \Omega_3 = -\frac{1}{2}\dot{\gamma} = \omega_{21}$$

✓ just as claimed in the lecture ☺