§... Parallel flows

N. St. eqns are very complicated because of the non lin. terms.

In certain situations these terms disappear, e.g. in parallel flows:

Def: In parallel flows the velocity is unidirectional

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w.l.o.g. align flow with the x-direction:

$$\underline{u}(x,y,z,t) = u(x,y,z,t)\,\underline{e_x}$$

Consequences?

$u = u(\cancel{x}, y, z, t) \quad = u(y, z, t)$

$v = w = 0$

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{w\frac{\partial v}{\partial z}} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{v\frac{\partial w}{\partial y}} + \cancel{w\frac{\partial w}{\partial z}} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\cancel{\frac{\partial^2 w}{\partial x^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}}\right)$$

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0. \qquad \frac{\partial u}{\partial x} = 0$$

where $\nu = \frac{\mu}{\rho}$ = kinematic viscosity

$$\boxed{\begin{aligned} \rho \frac{\partial u}{\partial t} &= \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ \frac{\partial p}{\partial y} &= \rho f_y \\ \frac{\partial p}{\partial z} &= \rho f_z \end{aligned}}$$

These are linear!!

<u>Special case: No body force $\underline{f} = 0$</u>

$p(x, \cancel{y}, \cancel{z}, t)$ but because of

$\Rightarrow p = p(x,t)$ into 1^{st} mom. eqn:

$$\underbrace{\rho \frac{\partial u}{\partial t}}_{\text{fct of } (y,z,t)} = \underbrace{-\frac{\partial p}{\partial x}}_{\underbrace{\text{fct of } (x,t)}_{\text{cannot depend on } x.}} + \underbrace{\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\text{fct. of } (y,z,t)}$$

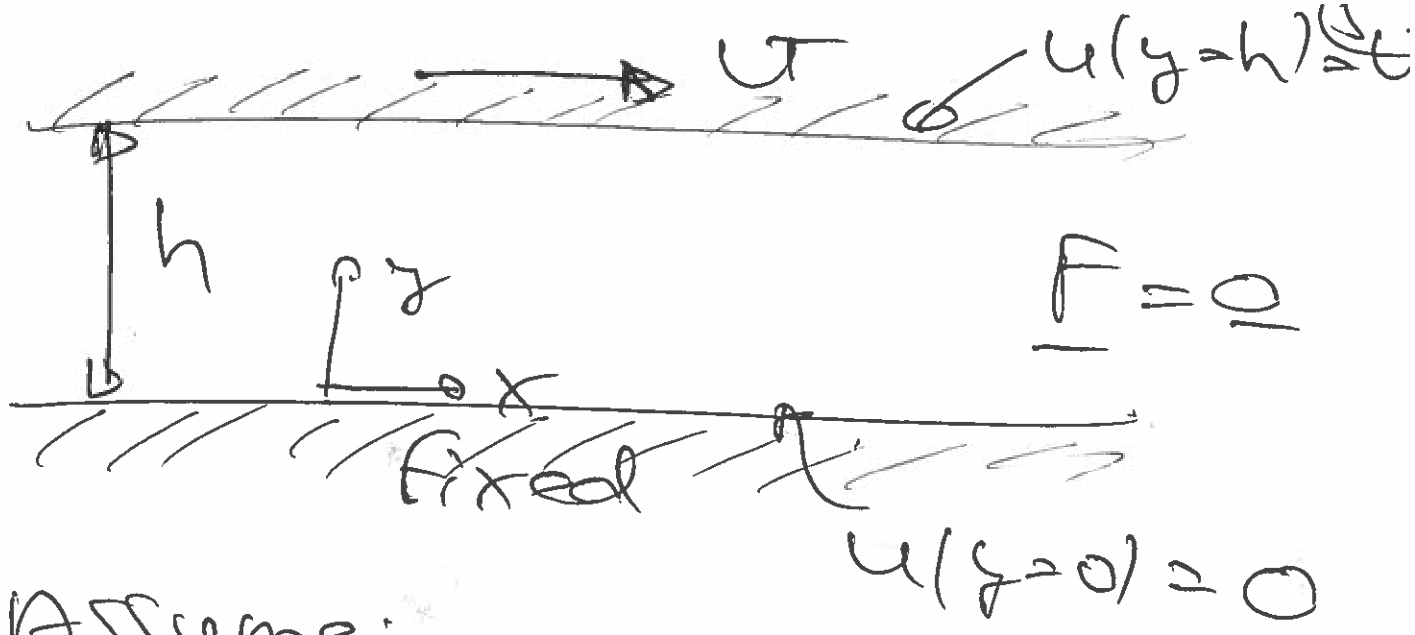
$$\frac{\partial p}{\partial x} = G(t)$$

Parallel flow eqns without body force

$$\boxed{\begin{gathered} \frac{\partial u}{\partial t} = -\frac{G}{\rho} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ \frac{\partial p}{\partial x} = G(t) \\ v = w = 0 \end{gathered}}$$

Example: Couette flow

Flow between parallel infinite plates. Upper plane moves with veloc U (in x-direction) parallel to itself.



$\underline{F} = \underline{0}$

Assume:

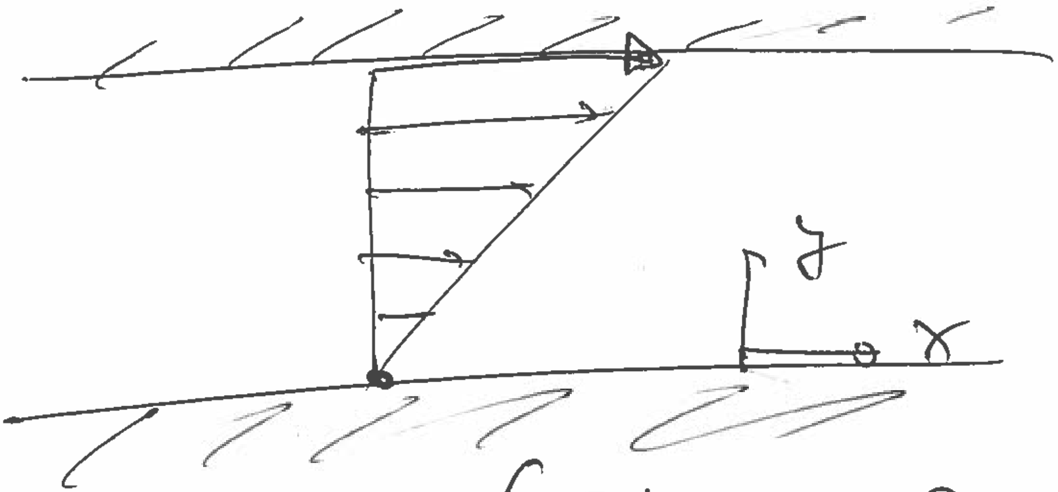
- parallel flow in x-direction
- flow is steady $\frac{\partial}{\partial t} = 0$
- $\frac{\partial u}{\partial z} = 0$
- $G = 0$

$$\cancel{\rho \frac{\partial u}{\partial t}} = -\cancel{G} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\boxed{0 = \frac{d^2 u}{dy^2}} \Rightarrow u(y) = Ay + B$$

Apply BCs:

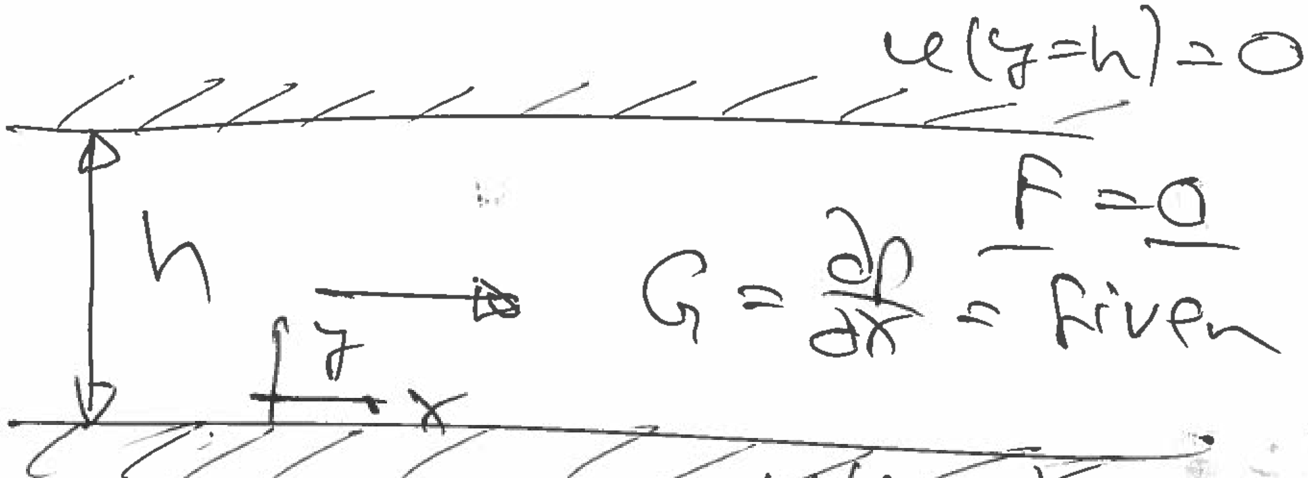
$$u(y) = U\left(\frac{y}{h}\right)$$



(Shear flow)

Example: Poisseuille flow

Pressure driven flow through a rigid channel



Assume

- parallel flow
- $\frac{\partial}{\partial t}=0$
- $\frac{\partial}{\partial z}=0$

$\left.\right\}\ u(y,z,t)=u(y)$

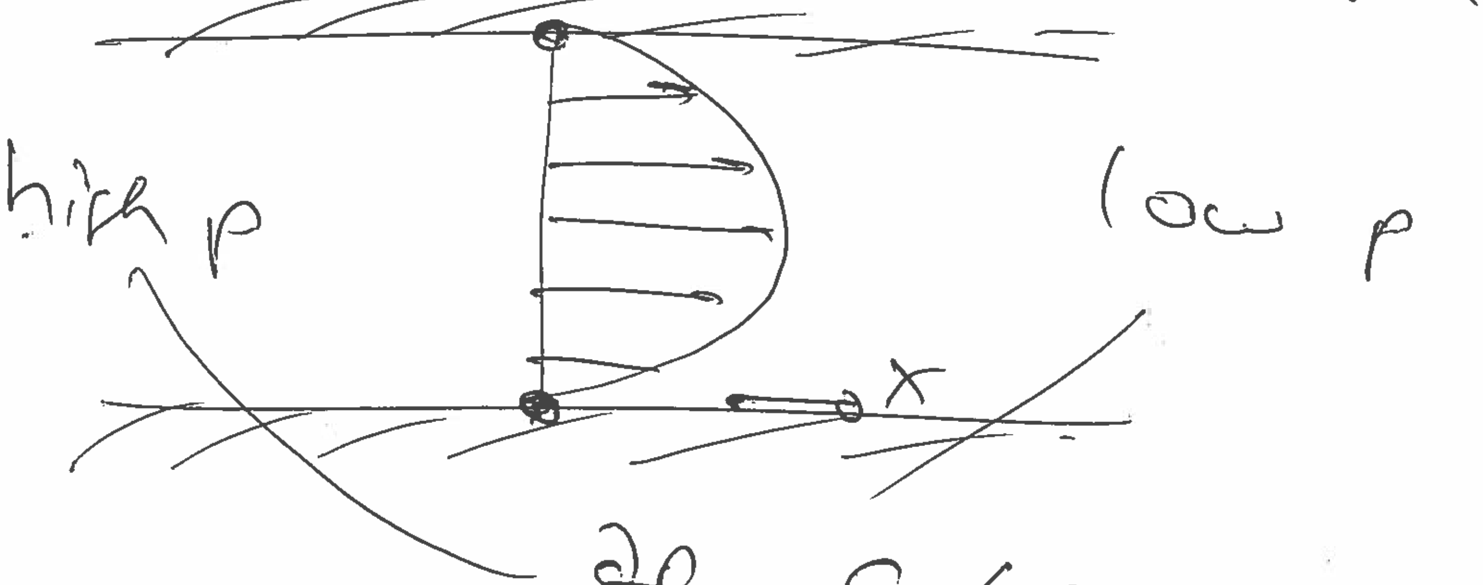
$$\cancel{\rho \frac{\partial u}{\partial t}} = -G + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\boxed{G = \mu \frac{d^2 u}{dy^2}} \Rightarrow$$

$$u(y) = \frac{1}{2}\frac{G}{\mu} y^2 + Ay + B$$

BC: $u(y=0) = u(y=h) = 0$

$$u(y) = \frac{G}{2\mu} \underbrace{\left(y^2 - hy\right)}_{<0 \text{ in } 0<y<h}$$



$$\frac{\partial p}{\partial x} = G < 0$$