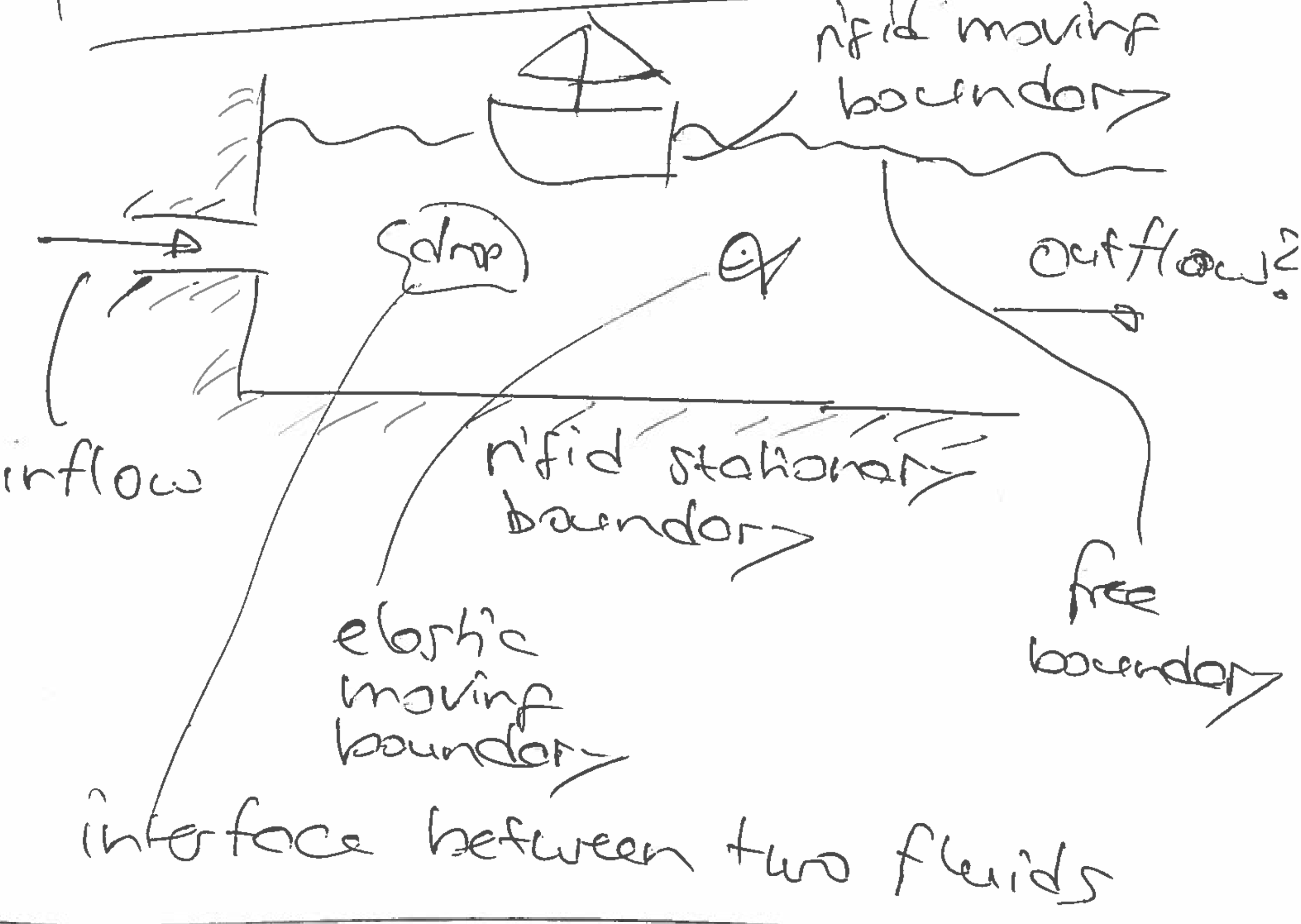
$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} \quad (1$$

$$\frac{\partial u_i}{\partial x_i} = 0$$



Initial conditions

Need to specify

$$u_i(x_j, t=0) = u_i^{(0)}(x_j)$$

Note: No I.C. for pressure!

Boundary conditions

(i) Inflow/outflow BC

$\boxed{u_i = v_i}$ = given, prescribed

(ii) on solid surfaces

"no slip & no penetration"

"solid veloc = fluid veloc"

given

$\boxed{u_i = v_i}$

given solid veloc.

Special case: stationary solid walls:

$\boxed{u_i = 0}$

(iii) free surfaces

We need 2 conditions:

- traction boundary condition
- Kinematic BC

(a) Kinematic BC

The posn. of the free surface can always be described implicitly as

$$f(x,y,z,t) = 0$$

or in 2D

$$f(x,y,t) = 0$$

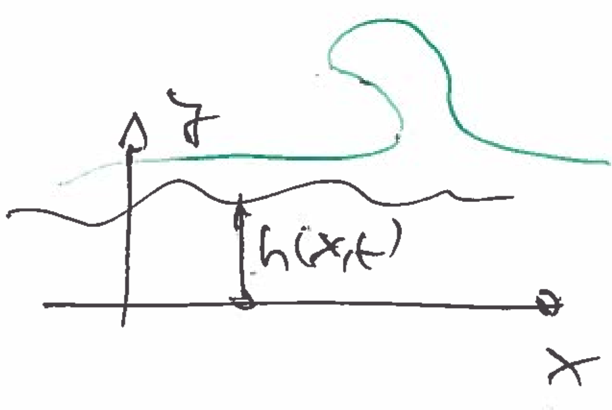
At least locally this can be inverted:

won't work.

$$z = h(x,y,t)$$

or in 2D

$$y = h(x,t)$$

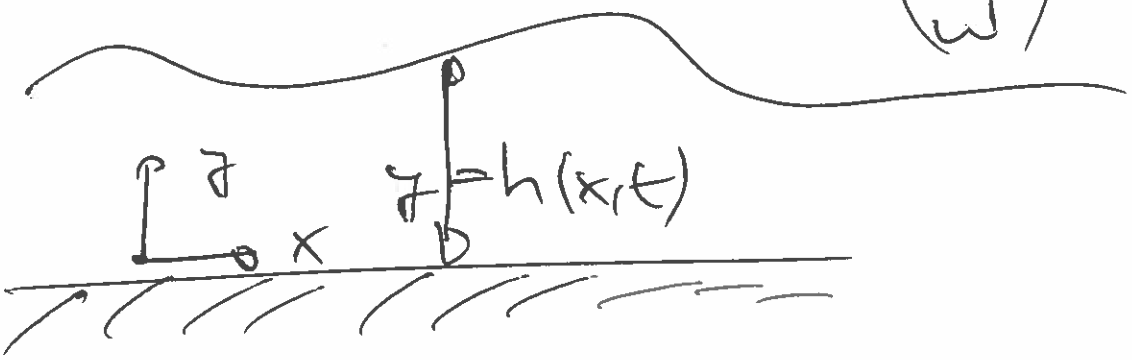


Physics: Fluid particles of the free surface stay of the free surface $\Rightarrow F \equiv 0$ for these particles.

$$\Rightarrow \boxed{\frac{DF}{Dt} = 0}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \begin{pmatrix} u \\ v \\ w \end{pmatrix} \leftrightarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

<u>Example:</u>



$$F(x,y,t) = 0$$

$$F(x,y,t) = h(x,t) - y$$

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0$$

$$\frac{\partial F}{\partial t} = \frac{\partial h}{\partial t}; \quad \frac{\partial F}{\partial x} = \frac{\partial h}{\partial x}; \quad \frac{\partial F}{\partial y} = -1$$

5

$$\boxed{\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - v = 0}$$

Links veloc @ surface with $h(x,t)$ & its derivs.

Special case 1:

Only vertical velocity, $u=0$

$$\frac{\partial h}{\partial t} + \cancel{u\frac{\partial h}{\partial x}} = v$$

$$\frac{\partial h}{\partial t} = v \quad \text{at } y=h$$

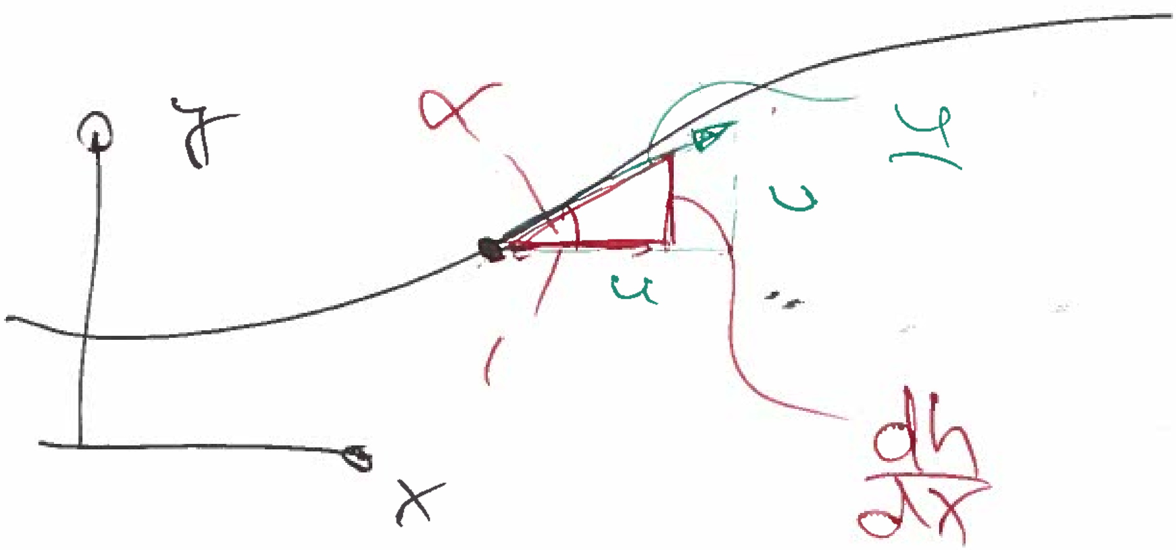
Special case 2:

Fixed free surface posn:

$$\frac{\partial h}{\partial t} = 0$$

$$\cancel{\frac{\partial h}{\partial t}} + u \frac{\partial h}{\partial x} = v \quad \text{at } y = h$$

$$\frac{\partial h}{\partial x} = \frac{dh}{dx} = \frac{v}{u} = \tan\alpha$$

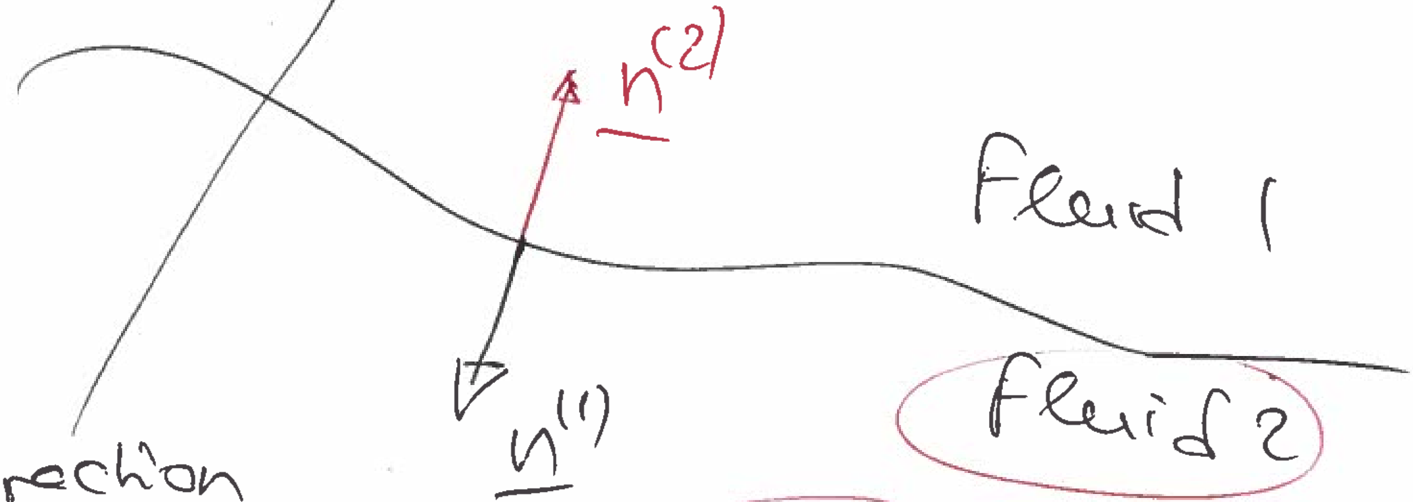


⇒ veloc $\underline{u}$ must be tangential to the surface!

(b) traction BC.

Physics: Stress must be continuous across the free surface (apart from surface tension)

$$t_i^{(1)} = \tau_{ij}^{(1)} n_j^{(1)}$$



traction acting **on** fluid 1.

$$t_i^{(2)} = \tau_{ij}^{(2)} n_j^{(2)}$$

By Newton's 3rd law:
"action = reaction"

$$\underline{t}^{(1)} = -\underline{t}^{(2)}$$

From geometry:

$$\underline{n}^{(1)} = -\underline{n}^{(2)}$$

$$\boxed{\tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j}$$

same (one of the two) outer unit normals.

Example: Hydrostatic stress

$$\tau_{ij} = -p\,\delta_{ij}$$

$$-p^{(1)} \delta_{ij} n_j = -p^{(2)} \delta_{ij} n_j$$

$$-p^{(1)} n_i = -p^{(2)} n_i$$

$$-p^{(1)} \underline{n} = -p^{(2)} \underline{n}$$

$$\underbrace{\left(p^{(1)} - p^{(2)}\right)}_{p^{(1)} = p^{(2)}} \underline{n} = \underline{0}$$