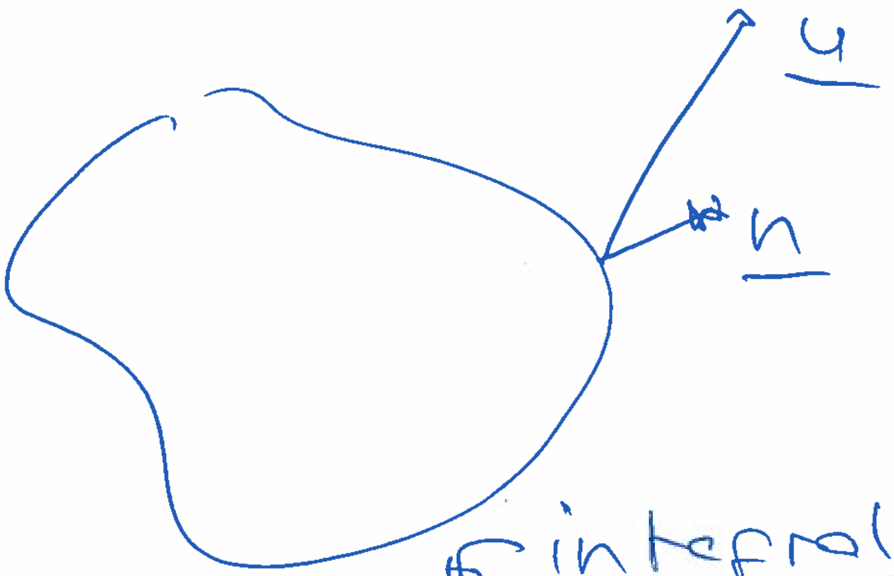


integral form

$$\boxed{-\oint (\rho \underline{u})\cdot \underline{n}\, dA = \int \frac{\partial \rho}{\partial t}\, dV}$$

density $\left[\frac{kg}{m^3}\right]$

$$\int \left(\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \underline{u}) \right) dV = 0$$

for any fixed volume

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \underline{u}) = 0}$$

differential form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$$\underbrace{\frac{D\rho}{Dt}} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

rate of change in density experienced by a fluid particle.

If the fluid is incompressible then $\frac{D\rho}{Dt} = 0$.

In that case the flow must satisfy

$$\boxed{\frac{\partial u_i}{\partial x_i} = 0 = \nabla \cdot \underline{u} = \operatorname{div} \underline{u}}$$

purely kinematic constraint!

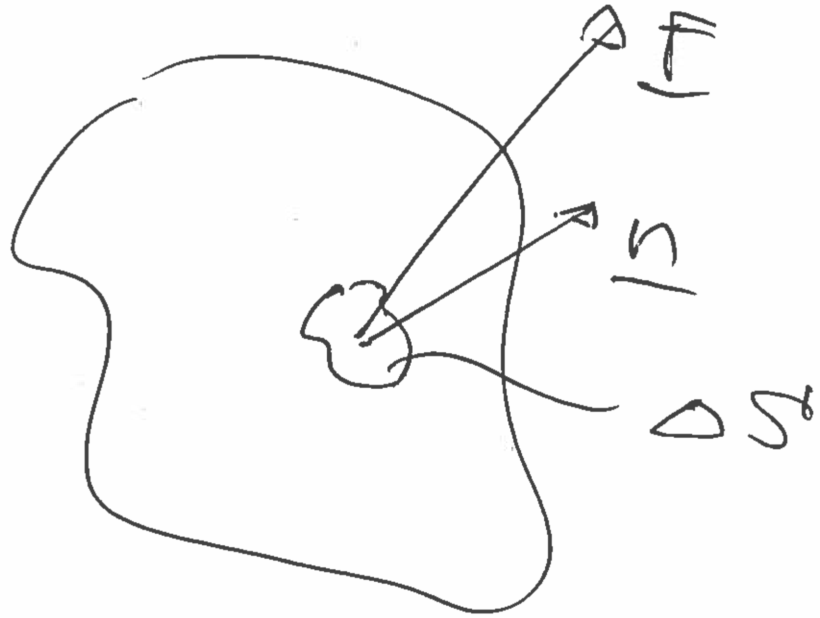
§3 Stress, Cauchy's eqn & the Navier-Stokes eqns

So far: Kinematics of flow.

Now: Look at forces on fluid particles & apply Newton's law to derive the equations of motion.

① The concept of stress/traction

Consider a blob of fluid loaded by some distributed force (pressure, shear stress)



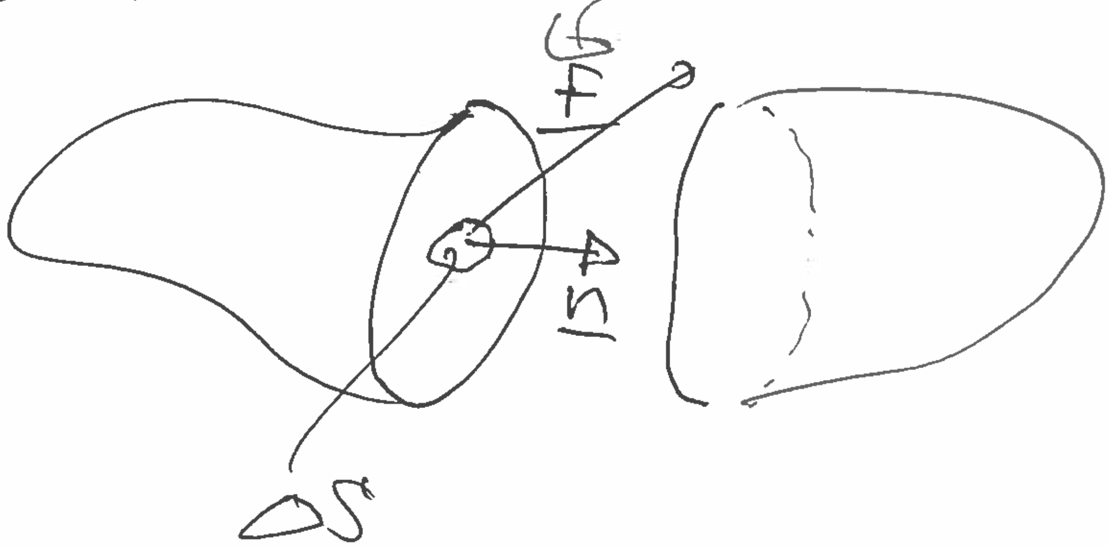
Patch ΔS on surface with outer unit normal $\underline{n}$ is subject to a resultant force $\underline{F}$.

Def: Traction

$$\underline{t} = \lim_{\Delta S \to 0} \frac{F}{\Delta S} \quad \text{vector!}$$

= force per unit area onto the fluid.

<u>Similarly:</u> Cut the blob along a plane with outer unit normal $\underline{n}$.



$\underline{F}$ is now the resultant force exerted onto ΔS by "the other half" of the fluid.

<u>Def:</u> Stress

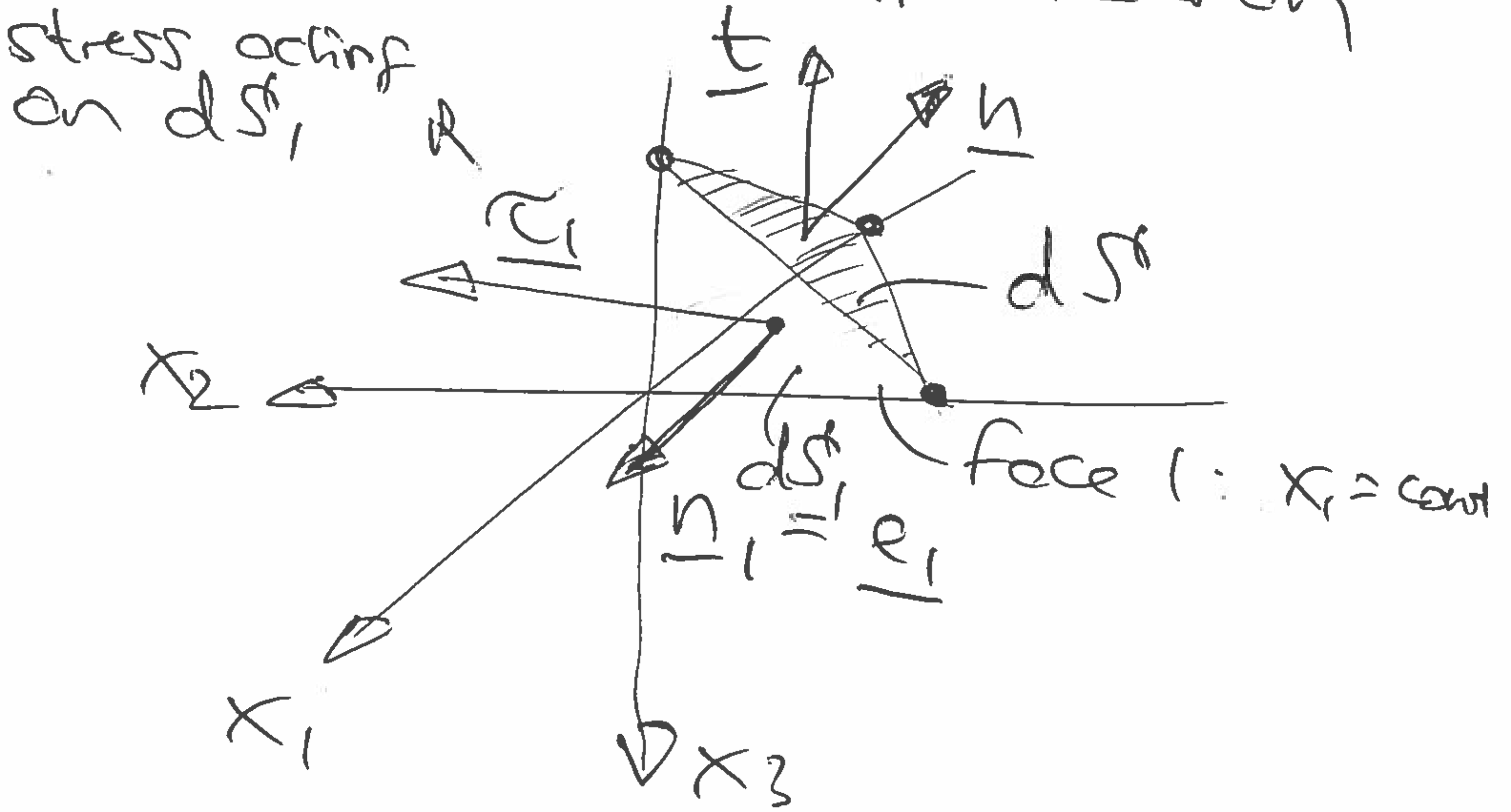
$$\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S} \quad \text{vector.}$$

Note: The stress will depend on

- the posn. in the fluid.
- direction of the imaginary cut, $\underline{n}$

2. The stress tensor

To examine dependence of $\underline{t}$ on $\underline{n}$ consider an infinitesimal tetrahedron



Represent faces (area & orientation) by an area vector $\parallel \underline{n}_i$ & has magnitude $d\sigma_i$.

Then: (Ex. sheet 3)

$$\underline{n}\, d\sigma + \underbrace{\underline{n}_i}_{\underline{e}_i} d\sigma_i = \underline{0} \qquad \Big| \cdot \underline{e}_j = \underline{n}_j$$

$$\underbrace{\underline{n}}_{\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}} \cdot \underline{e}_j\, d\sigma + \underbrace{\underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}} d\sigma_i}_{d\sigma_j} = 0$$

$$\underbrace{\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}}_{n_j}$$

$$\boxed{d\sigma_j = -n_j\, d\sigma}$$