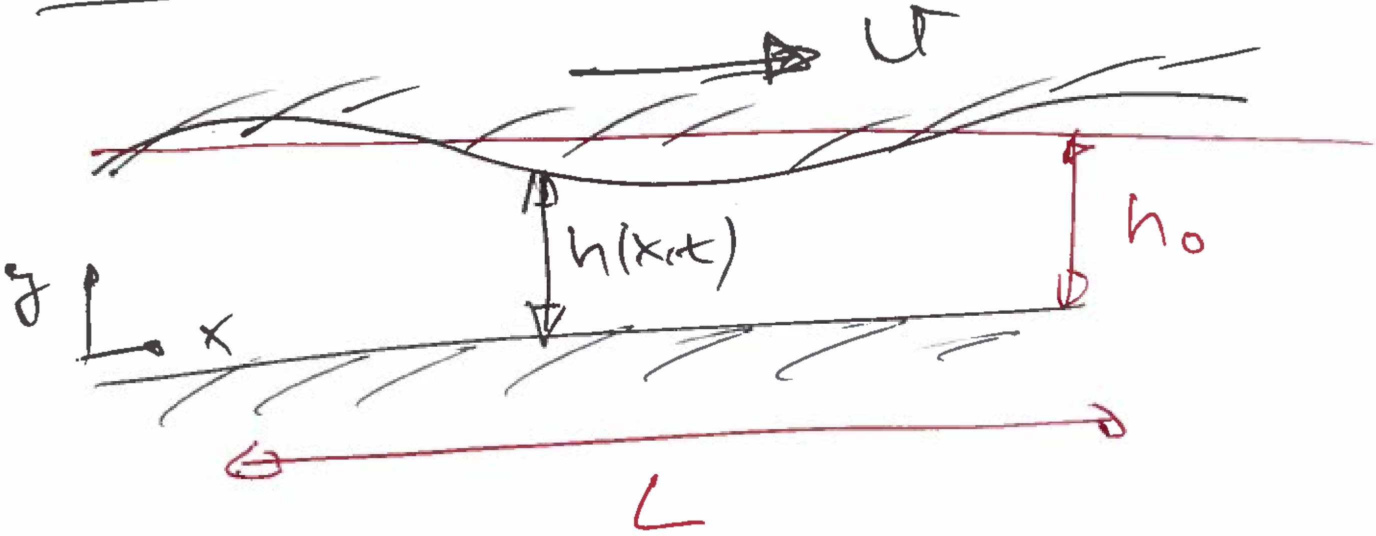
Lubrication theory



Narrow gap: $\frac{h_0}{L} \ll 1$

& gently varying $\left|\frac{\partial h}{\partial x}\right| = O\left(\frac{h_0}{L}\right) \ll 1$

Scale:

$x = L\tilde{x}$ $\qquad t = \frac{L}{U}\tilde{t}$

$y = h_0\tilde{y}$ $\qquad p = P\tilde{p}$

$u = U\tilde{u}$

$v = V\tilde{v}$

V & P unknown

Continuity: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ (2

$$\frac{U}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{V}{h_0}\frac{\partial \tilde{v}}{\partial \tilde{y}}=0$$

These terms balance if

$$V=\frac{h_0}{L}U \ll U$$

makes sense.

x-mom. eqn:

$$\rho\left(\frac{U^2}{L}\frac{\partial \tilde{u}}{\partial \tilde{t}}+\frac{U^2}{L}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\overbrace{\frac{U\cancel{h_0}U}{\cancel{h_0}L}}^{\frac{U^2}{L}}\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right)=$$

$$-\frac{P}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}}+\mu\left(\frac{U}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}+\frac{U}{h_0^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)$$

$$\frac{U^2\rho}{L}\frac{D\tilde{u}}{D\tilde{t}}=-\frac{P}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}}+\frac{\mu U}{h_0^2}\left(\cancel{\left(\frac{h_0}{L}\right)^2\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}}+\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)$$

All derivs. of quantities with tildes are $O(1)$

$$\underbrace{\underbrace{\frac{\rho U h_0}{\mu}}_{Re}\underbrace{\left(\frac{h_0}{L}\right)}_{\ll 1}}\frac{D\tilde{u}}{D\tilde{t}} = -\frac{P}{\left(\frac{\mu U}{h_0}\right)}\left(\frac{h_0}{L}\right)\frac{\partial\tilde{p}}{\partial\tilde{x}} + \frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}$$

Can neglect inertial terms if $Re\left(\frac{h_0}{L}\right) \ll 1$

The last two terms, which represent a balance between press. gradient & the viscous effects, balance if

$$P = \frac{\mu U}{h_0}\underbrace{\left(\frac{L}{h_0}\right)}_{\gg 1}$$

The pressure is much larger than the typical shear stress

In that case:

$$\frac{\partial \tilde{p}}{\partial \tilde{x}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

Back to dimensional terms:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

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y-comp. of NSE:

Ex. Sheet ? (?):

To leading order:

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0 \; ; \quad \frac{\partial p}{\partial y} = 0$$

So the ~~eqns are~~

$$\boxed{\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad (1) \\ 0 &= -\frac{\partial p}{\partial y} \qquad (2) \end{aligned}}$$

Parallel flow eqns.

Because of the gentle slope of the wall the flow "sees" locally parallel flow.

BC: $u(y=0) = 0$

$$u(y=h(x,t)) = U$$

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(2) : $p(x,\cancel{y}) = p(x)$

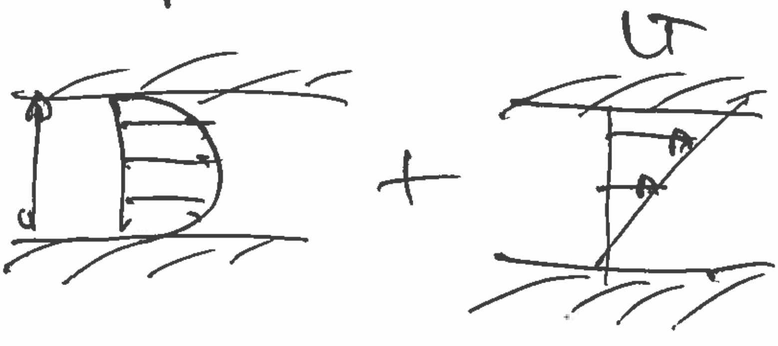
(1) $$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + Ay + B$$

Apply BC:

$$u(x,y,t) = \underbrace{\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h(x,t)y\right)}_{\text{press. driven flow}} + \underbrace{U\frac{y}{h(x,t)}}_{\text{shear flow}}$$

h(x,t)



Shear flow

BUT: what is $\frac{dP}{dx}$?