1

$$\boxed{u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}}$$

$$\boxed{\frac{D\underline{\omega}}{Dt} = \underbrace{(\underline{\omega}\cdot\nabla)\underline{u}}_{\underline{0} \text{ in 2D; also}} + \nu \nabla^2 \underline{\omega}}$$

$$\underline{\omega} = \omega \underline{e_z}$$

Scale:

Length: $a$ ; veloc. scale: $U$

$$\underline{u} = U\,\underline{\tilde{u}}^*$$

$$\underline{r} = a\,\underline{\tilde{r}}^*$$

$$t = \frac{a}{U}\tilde{t}$$

$$\underline{\omega} = \nabla\times\underline{u} = \frac{U}{a}\underline{\tilde{\omega}}$$

$$\frac{U^2}{a^2}\frac{\partial \tilde{\omega}_i}{\partial \tilde{t}} + \frac{U^2}{a^2}\tilde{u}_j\frac{\partial \tilde{\omega}_i}{\partial \tilde{x}_j} = \frac{U^2}{a^2}\tilde{\omega}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{U}{a^3}\nu\frac{\partial^2 \tilde{\omega}_i}{\partial \tilde{x}_j^2}$$

$$\overbrace{\frac{Ua}{\nu}}^{Re}\frac{D\tilde{\omega}}{D\tilde{t}} = \overbrace{\frac{Ua}{\nu}}^{Re}(\underline{\tilde{\omega}}\cdot\tilde{\nabla})\tilde{u} + \tilde{\nabla}^2\underline{\tilde{\omega}}$$

As $Re \to 0$:

$$\tilde{\nabla}^2 \underline{\tilde{\omega}} = \underline{0}\ ; \quad \nabla^2 \underline{\omega} = \underline{0}$$

Aside: Start with Stokes eqns:

$$\nabla p = \mu \nabla^2 \underline{u}$$

Take curl:

$$\underbrace{\nabla \times \nabla p}_{\underline{0}} = \mu \nabla^2 \underbrace{\nabla \times \underline{u}}_{\underline{\omega}}$$

$$\nabla^2 \underline{\omega} = \underline{0}$$

Now consider 2D flow:

$$\underline{\omega} = \omega \underline{e_z}$$

$$\nabla^2 \omega = 0 \quad \text{also} \quad \omega = -\nabla^2 \psi$$

2D Stokes flow

$$\Rightarrow \boxed{\nabla^2 \nabla^2 \psi = 0}$$

$$\nabla^4 \psi = 0$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

one scalar 4th order linear PDE for $\psi(x,y)$

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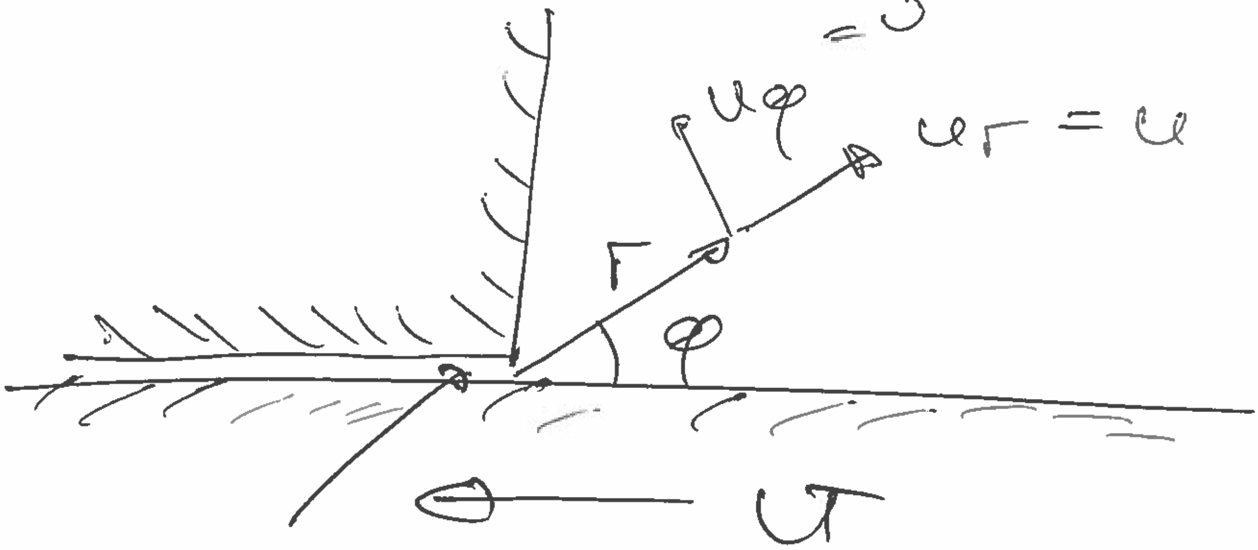
**Example:**

Scraping flow




zero gap width.

Assume: slow, viscous flow

$\Rightarrow$ Stokes eqns are valid since $Re \ll 1$

$\nabla^4 \psi = 0$

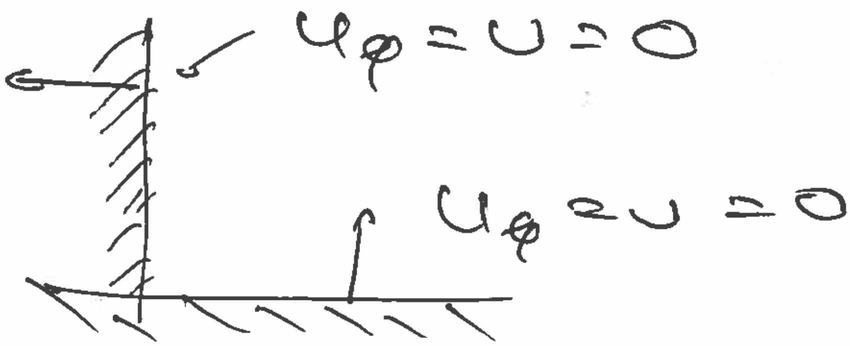
Use polar coords.

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

$$u = u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi}$$

$$v = u_\varphi = -\frac{\partial \psi}{\partial r}$$

<u>BC:</u> Impermeability:




So:

$$v = -\frac{\partial \psi}{\partial r} = 0 \quad \text{at } \varphi = 0 \ \forall r$$

$$\psi(\varphi = 0) = \text{const} = C_1$$

<u>Also:</u>

$$v = -\frac{\partial \psi}{\partial r} = 0 \quad \text{at } \varphi = \frac{\pi}{2} \ \forall r$$

$$\psi(\varphi = \pi/2) = \text{const} = C_2$$

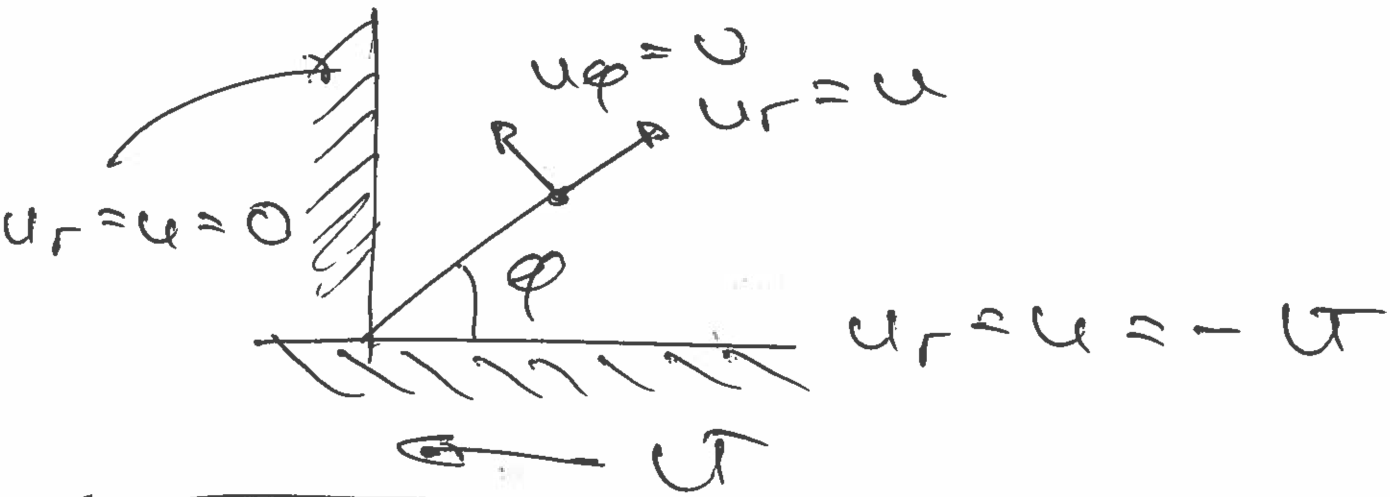
<u>Note:</u> zero flux into corner $\Rightarrow \psi$ continuous

$\Rightarrow \quad C_1 = C_2 = C = 0$

w.l.o.g. for $\underline{u}$

$$\boxed{\begin{array}{ll} \psi(\varphi = 0) = 0 & (1) \\ \psi(\varphi = \frac{\pi}{2}) = 0 & (2) \end{array}}$$

<u>No slip:</u>




$$\boxed{\begin{array}{lll} u = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} & \text{at } \varphi = \frac{\pi}{2} & (3) \\ u = -U = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} & \text{at } \varphi = 0 & (4) \end{array}}$$

want $\psi(r, \varphi)$

Try: $\psi(r,\varphi) = g(r)\, f(\varphi)$

Now BC (3) & (4) show that $\frac{1}{r}\frac{\partial \psi}{\partial \varphi}$ should be indep. of $r$. (at least at $\varphi = 0$, $\varphi = \pi/2$)

This suggest to try

$$g(r) = U r$$

Ansatz: $\boxed{\psi(r,\varphi) = U r\, f(\varphi)}$

BC for $f(\varphi)$:

$\varphi = 0: \quad \psi = 0$

$\varphi = \frac{\pi}{2}: \quad \psi = 0$

$\varphi = 0: \quad \frac{1}{r}\frac{\partial \psi}{\partial \varphi} = -U$

$\varphi = \frac{\pi}{2}: \quad \frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0$

$$\boxed{\begin{aligned} f(0) &= 0 \\ f\left(\tfrac{\pi}{2}\right) &= 0 \\ f'(0) &= -1 \\ f'\left(\tfrac{\pi}{2}\right) &= 0 \end{aligned}}$$

PDE: $\nabla^2\nabla^2\psi = 0$ (7

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) \sigma r f(\varphi)$$

$$= \frac{\sigma}{r} f + \frac{\sigma}{r} f'' = \sigma r^{-1}(f + f'')$$

$$\nabla^2\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) \sigma r^{-1}(f + f'')$$

$$= \sigma \left\{ (-1)(-2) r^{-3} (\underline{\underline{f}} + \underset{\sim}{f''}) + (-1) r^{-3} (\underline{\underline{f}} + \underset{\sim}{f''}) + r^{-3} (\underset{\sim}{f''} + f^{IV}) \right\}$$

$$= \frac{\sigma}{r^3} \left\{ (2-1) f + (2-1+1) f'' + f^{IV} \right\}$$

$$\nabla^4\psi = \frac{\sigma}{r^3} \underbrace{\left( f^{IV} + 2f'' + f \right)}_{=0} = 0 \qquad \forall r, \varphi$$

4th order, linear, homog, const. coeff

$f(\varphi) \sim e^{\lambda\varphi}$

char. poly:

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

$$(\lambda^2+1)^2 = 0$$

$\lambda_{1234} = \pm i$ repeated roots

$f(\varphi) =$ ~~[illegible]~~

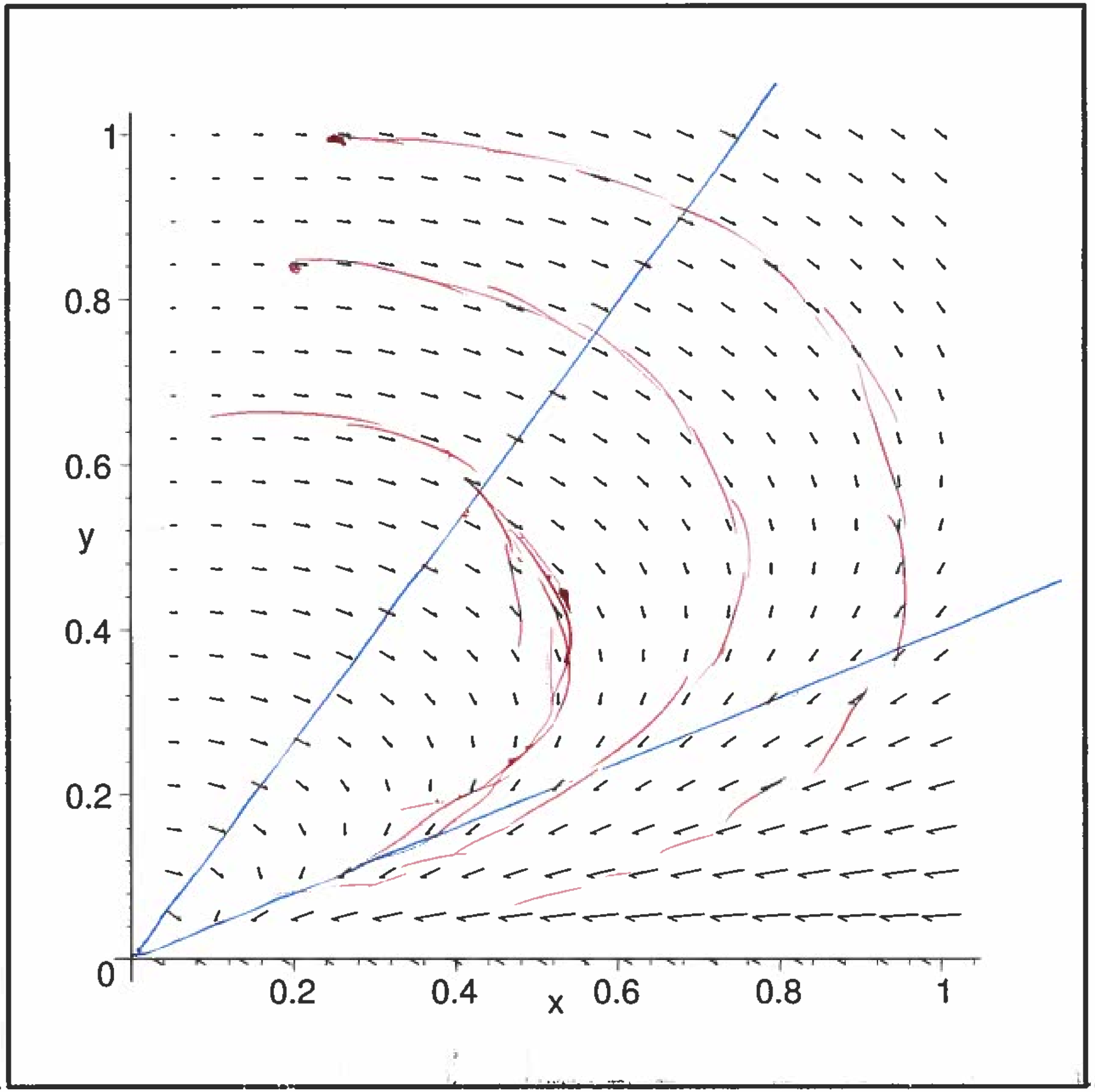
$$A\sin\varphi + B\cos\varphi + C\varphi\sin\varphi + D\varphi\cos\varphi$$

1st year!!!!

A, B, C, D from BC...

$$\psi = \frac{Ur}{\left(\frac{\pi}{2}\right)^2 - 1}\left(-\left(\frac{\pi}{2}\right)^2\sin\varphi + \varphi\cos\varphi + \frac{\pi}{2}\varphi\sin\varphi\right)$$

Velocity field for scraping flow at zero Reynolds number:
The vertical wall at x=0 is stationary. The horizontal wall at y=0 moves to the left with unit velocity.



So: $u_\varphi = v = -\frac{\partial \psi}{\partial r}$ is indep. of r

$u_r = u = \frac{1}{r}\frac{\partial \psi}{\partial \varphi}$

Discussion:

I Nonuniformity of the soln

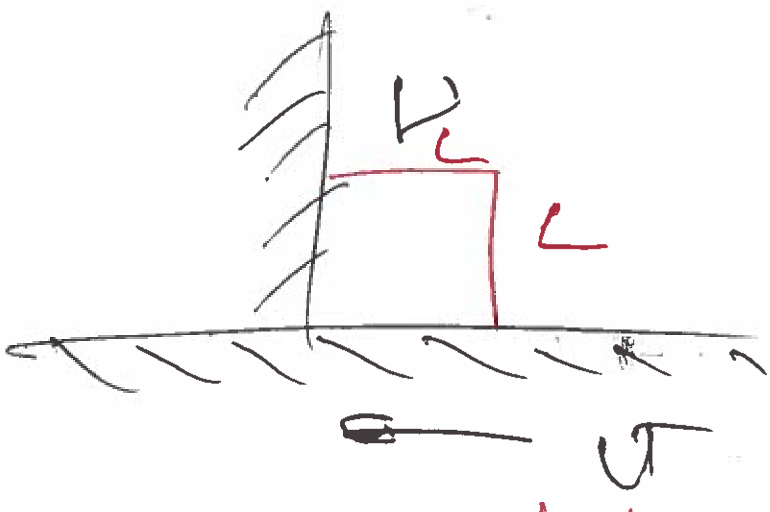
We have assumed $Re \ll 1$

$Re = \frac{U a}{\nu}$

U → veloc of bottom plate

ν → viscosity of fluid

what is a?




There is no lengthscale!

choose e.g. distance from corner

choose lengthscale $L = a$ !!

If $a$ is suff. small then $Re$ can become arbitrarily small BUT equally at large distances $Re$ becomes arbitrarily large

$\Rightarrow$ Stokes eqns invalid.

$\Rightarrow$ $\psi$ form is only valid locally.

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Need: $Re = \frac{Ua}{\nu} \ll 1$

$$\Rightarrow \boxed{a \ll \frac{\nu}{U}}$$