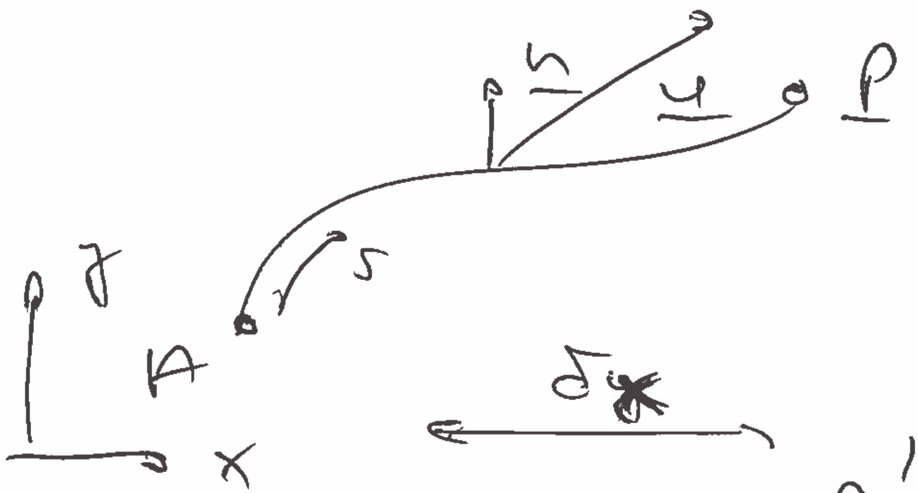
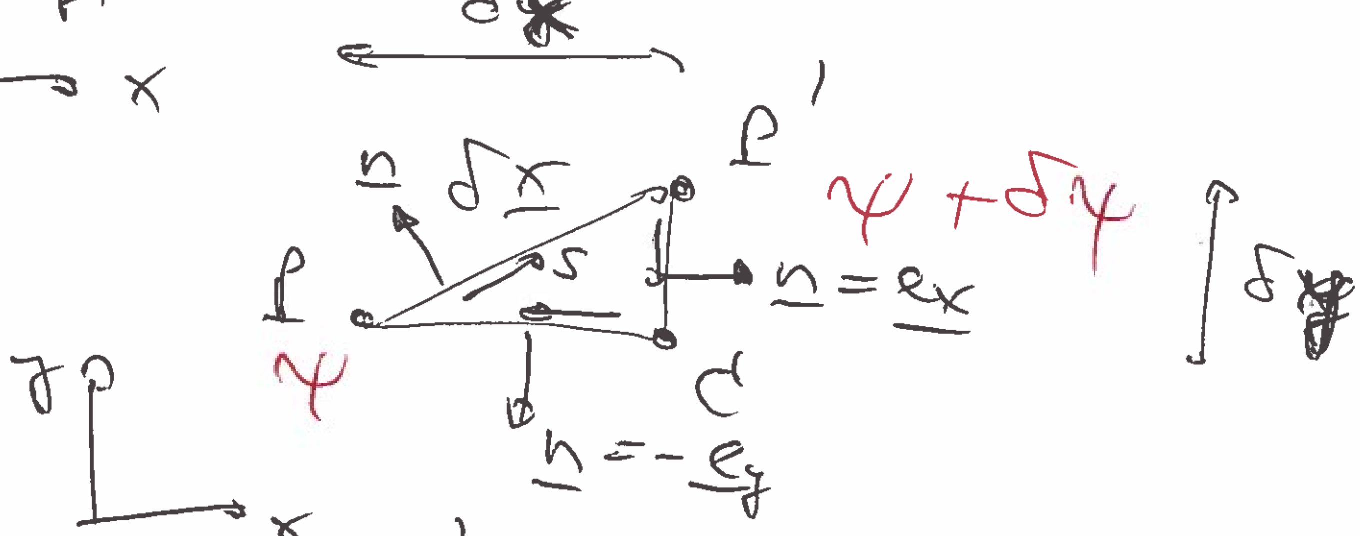
(1

$$\psi_A(\underline{P}) = \int_A^P \underline{u}\cdot\underline{n}\,ds$$

to left





$$\delta\psi = \int_P^{P'} \underline{u}\cdot\underline{n}\,ds$$

$$\oint_{\triangle} \underline{u}\cdot\underline{n}\,ds = 0$$

$$\delta\psi = \int_P^{P'} \underline{u}\cdot\underline{n}\,ds = -\int_{P'}^{C} \underbrace{\underline{u}\cdot\underline{n}}_{u}\,\overbrace{ds}^{-dy} - \int_C^{P} \underbrace{\underline{u}\cdot\underline{n}}_{-v}\,\overbrace{ds}^{-dx}$$

$\delta x, \delta y \to 0$

+ mean value theorem.

$$\delta\psi = u\,\delta y - v\,\delta x$$

Also: $\psi(x,y)$

$$\delta\psi = \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y$$

$$\boxed{\begin{aligned}\frac{\partial\psi}{\partial x} &= -v \\ \frac{\partial\psi}{\partial y} &= u\end{aligned}}$$

Similar to Cauchy-Riemann, potential, Airy stress fct., etc.

Remarks:

(1) Derivation involved continuity eqn. $\Rightarrow$ continuity eqn. should be satisfied automatically:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \overset{?}{=} 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \overset{?}{=} 0$$

(2) Relation to vorticity: (2D)

$$\underline{\omega} = \omega_z \underline{e_z} = \omega \underline{e_z} = \nabla \times \underline{u}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\omega = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\boxed{\omega = -\nabla^2 \psi}$$

The Vorticity equation

- Use of the streamfunction allows us to automatically satisfy the continuity equation. Now we will try to transform the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

into an equation for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

- Here are a few results that we will use in the derivation:

$$\frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}), \tag{1}$$

$$\nabla \times \nabla \phi = 0, \tag{2}$$

and

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} + \mathbf{u}\underbrace{\nabla \cdot \boldsymbol{\omega}}_{0} - \boldsymbol{\omega}\underbrace{\nabla \cdot \mathbf{u}}_{0}, \tag{3}$$

where the last two terms vanish because

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u}) = \operatorname{div} \operatorname{curl} \mathbf{u} = 0$$

and

$$\nabla \cdot \mathbf{u} = 0.$$

- First, we use (1) in the momentum equation to obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$$

i.e.

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\boldsymbol{\omega}}_{LHS} = \underbrace{-\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}}_{RHS}$$

- Now take the curl of the LHS:

$$\nabla\times LHS = \frac{\partial}{\partial t}\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}} + \frac{1}{2}\underbrace{\nabla\times\nabla(\mathbf{u}\cdot\mathbf{u})}_{0\text{ because of (2)}} - \underbrace{\nabla\times(\mathbf{u}\times\boldsymbol{\omega})}_{\text{see (3)}}$$

i.e.

$$\nabla\times LHS = \frac{\partial\boldsymbol{\omega}}{\partial t} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$$

- ...and the RHS:

$$\nabla\times RHS = -\frac{1}{\rho}\underbrace{\nabla\times\nabla p}_{0\text{ because of (2)}} + \nu\nabla^2\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}}$$

- Now combine the remaining non-zero terms

$$\underbrace{\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}}_{D\boldsymbol{\omega}/Dt} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = \nu\nabla^2\boldsymbol{\omega}.$$

- The resulting equation is the *vorticity transport equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \tag{4}$$

  which shows that the rate of change of the vorticity of material particles, $D\boldsymbol{\omega}/Dt$, is controlled by 'vortex stretching' (described by $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$; this is a familiar result from inviscid fluid mechanics) and by diffusion (described by $\nu\nabla^2\boldsymbol{\omega}$). The diffusion of vorticity only occurs in viscous flows.

- For 3D flows, the first term on the RHS in (4) represents vortex stretching: velocity gradients lead to a change in the rate of rotation of material particles.

- Note that for 2D flows, vortex stretching is absent since $\mathbf{u} = u(x, y)\ \mathbf{e}_x + v(x, y)\ \mathbf{e}_y$ and $\boldsymbol{\omega} = \omega(x, y)\ \mathbf{e}_z$ and therefore $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0$.

- The vorticity transport equation provides an interesting interpretation of the kinematic viscosity ν: The kinematic viscosity is the diffusion coefficient for the diffusion of vorticity.

- Many phenomena in viscous fluid mechanics can be interpreted in terms of the diffusion of vorticity but this is (unfortunately) beyond the scope of this course.

- For 2D flows, the vorticity transport equation

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega$$

together with the equation for the vorticity in terms of the streamfunction

$$\omega = -\nabla^2 \psi$$

and

$$u = \partial\psi/\partial y \quad \text{and} \quad v = -\partial\psi/\partial x$$

provide the streamfunction-vorticity formulation of the Navier-Stokes equations, which consists of only two PDEs for the scalars ω and ψ rather than the three equations for u, v and p in the 'primitive variable' form.

- Scaling arguments show that in the limit of zero Reynolds number, only one fourth-order PDE for the streamfunction ψ needs to be solved, namely the biharmonic equation

$$\nabla^4 \psi = 0,$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

- This can also be shown directly by taking the curl of the Stokes equations.