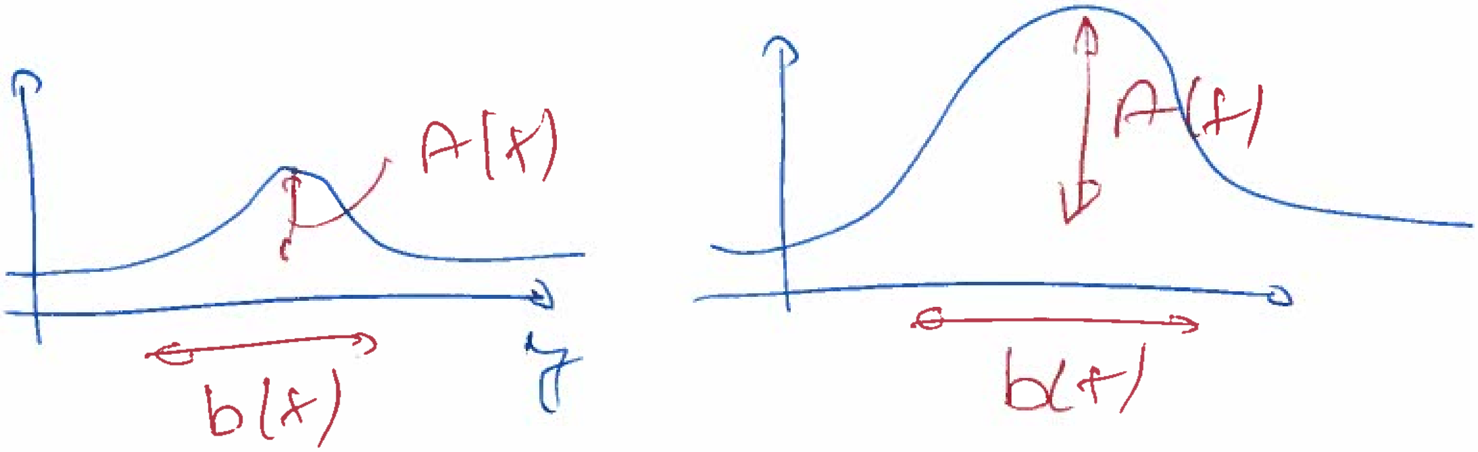
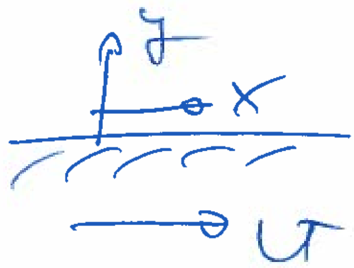
Similarity solns

$$u(y,t) = A(t)\, f\left(\frac{y}{b(t)}\right)$$



$\underline{u} = u(y,t)\,\underline{e}_x$



$$\boxed{\begin{gathered}\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \\ \left.\begin{aligned} u &= U \text{ at } y = 0 \\ u &\to 0 \text{ as } y \to \infty \end{aligned}\right\} t > 0 \\ u = 0 \text{ at } t = 0\end{gathered}}$$

(Clever stuff: $u(y,t) = U f(\eta)$ $\qquad \eta = \dfrac{y}{\sqrt{\nu t}}$

2

$$\boxed{f'' + \frac{1}{2}\eta f' = 0}$$

$f(\eta = 0) = 1$

$f \to 0$ as $\eta \to \infty$ $(\times 2)$

$F = f'$

$$F' + \frac{1}{2}\eta F = 0$$

$$\frac{F'}{F} = -\frac{1}{2}\eta$$

$$\ln\frac{F}{F_0} = -\frac{1}{4}\eta^2$$

$$F = f' = F_0 \exp\left(-\frac{1}{4}\eta^2\right)$$

$$f(\eta) = A + F_0 \int^{\eta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

Lower limit? Arbitrary!

Choose ∞.

$$f(\eta) = A + F_0 \int_{\infty}^{\zeta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

$$= A + B \int_{\zeta}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

BC:

$$f(\zeta \to \infty) = 0 \Rightarrow A = 0$$

$$f(\zeta = 0) = 1 = B \underbrace{\int_{0}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi}_{\sqrt{\pi}}$$

$$B = \frac{1}{\sqrt{\pi}}$$

$$u(y,t) = \frac{U}{\sqrt{\pi}} \underbrace{\int_{\zeta}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi}_{\operatorname{erfc}\left(\frac{\zeta}{2}\right)}$$

$$\zeta = \frac{y}{\sqrt{\nu t}}$$

§... Stream function & vorticity eqns

Alternative formulation of the N.St. eqns. particularly powerful in 2D.

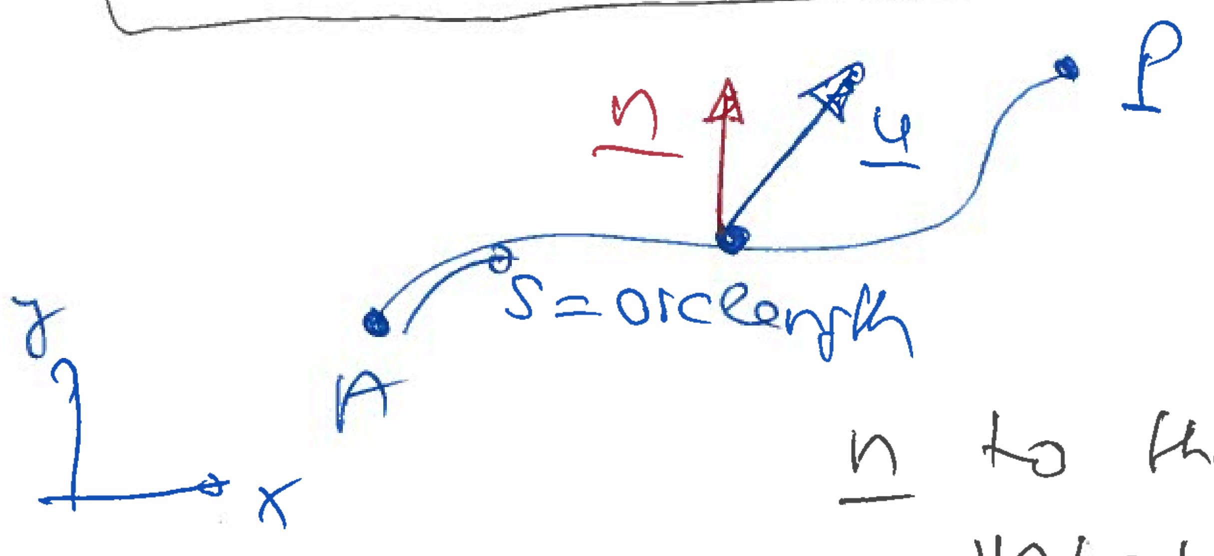
Stream fct.

(2D, incompr. flow)

$$\underline{u} = u\,\underline{e_x} + v\,\underline{e_y}$$

Def:

$$\boxed{\psi_A(P) = \int_A^P \underline{u} \cdot \underline{n}\, ds}$$



$\underline{n}$ to the left.

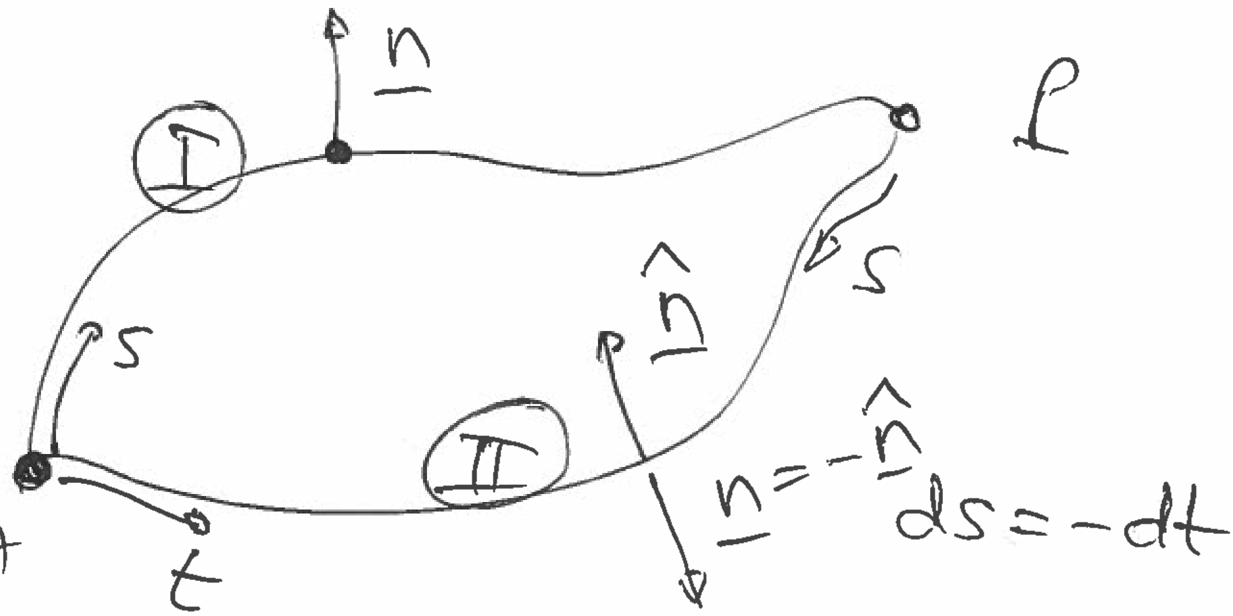
$|\underline{n}| = 1$

$\psi_A(P)$ represents the volume flux (per unit depth) crossing the line $A \to P$.

Implications:

(1) $\psi_A(P)$ is path indep.

Proof:



$$\psi_A^{\textcircled{I}}(P) = \int_{\textcircled{I}\,A}^{P} \underline{u} \cdot \underline{n}\, ds$$

$$\psi_A^{\textcircled{II}}(P) = \int_{\textcircled{II}\,A}^{P} \underline{u} \cdot \underbrace{\hat{\underline{n}}}_{-\underline{n}}\, \underbrace{dt}_{-ds}$$

6

$$= - \int_{P\ \mathrm{II}}^{A} \underline{u} \cdot \underline{n}\, ds$$

$$\psi_A^{\mathrm{I}}(P) - \psi_A^{\mathrm{II}}(P) = \int_{A\ \mathrm{I}}^{P} \underline{u} \cdot \underline{n}\, ds + \int_{P\ \mathrm{II}}^{A} \underline{u} \cdot \underline{n}\, ds$$

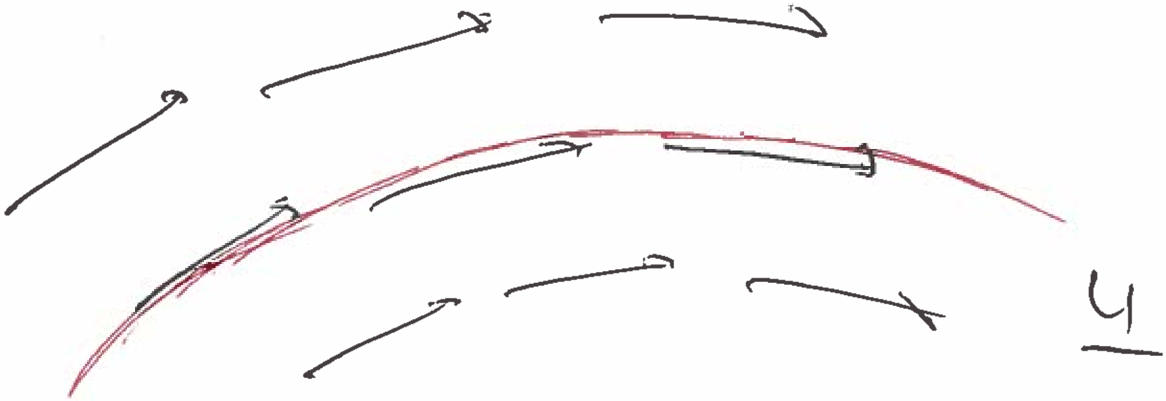
$$= \oint_{APA} \underline{u} \cdot \underline{n}\, ds = 0$$

because of the integral continuity eqn for incompr. fluids.

$\square$

(2) ψ is constant along streamlines.

↳ lines that are tangential to the veloc field.

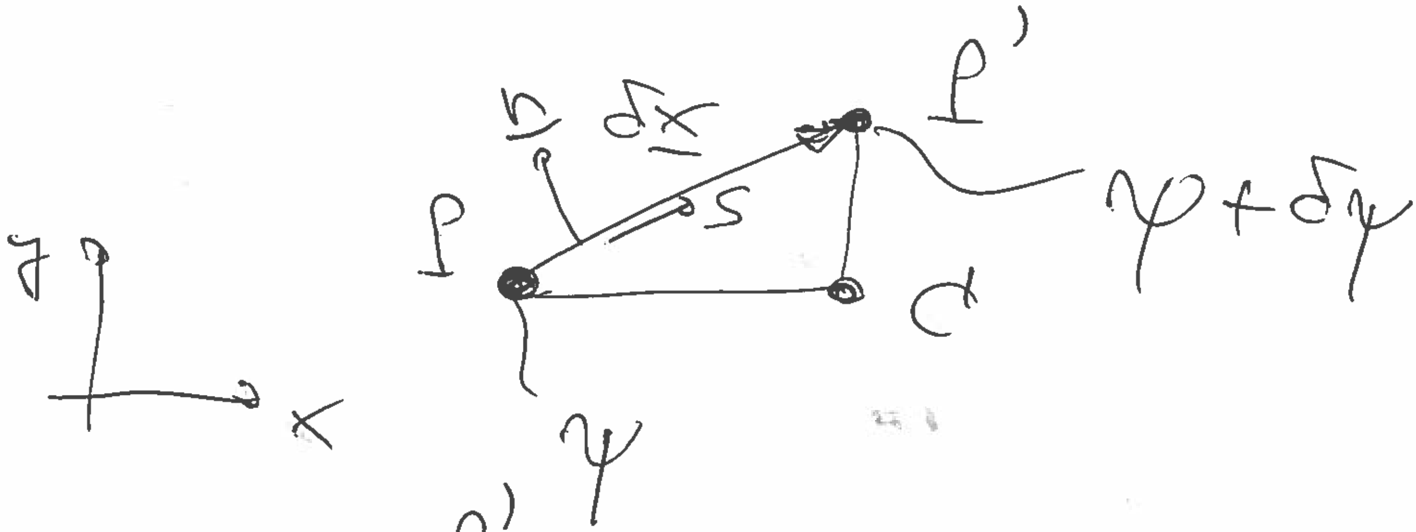


$\underline{u} \cdot \underline{n} = 0$ along streamlines

(3) impermeable boundaries are streamlines in this sense because $\underline{u} \cdot \underline{n} = 0$

<u>Convention:</u> Set $\psi = 0$ along impermeable boundaries.

what PDE does ψ satisfy?



$$\delta\psi = \int_{P}^{P'} \underline{u} \cdot \underline{n} \, ds$$

Integral continuity:

$$\oint_{\Delta} \underline{u} \cdot \underline{n} \, ds = 0$$