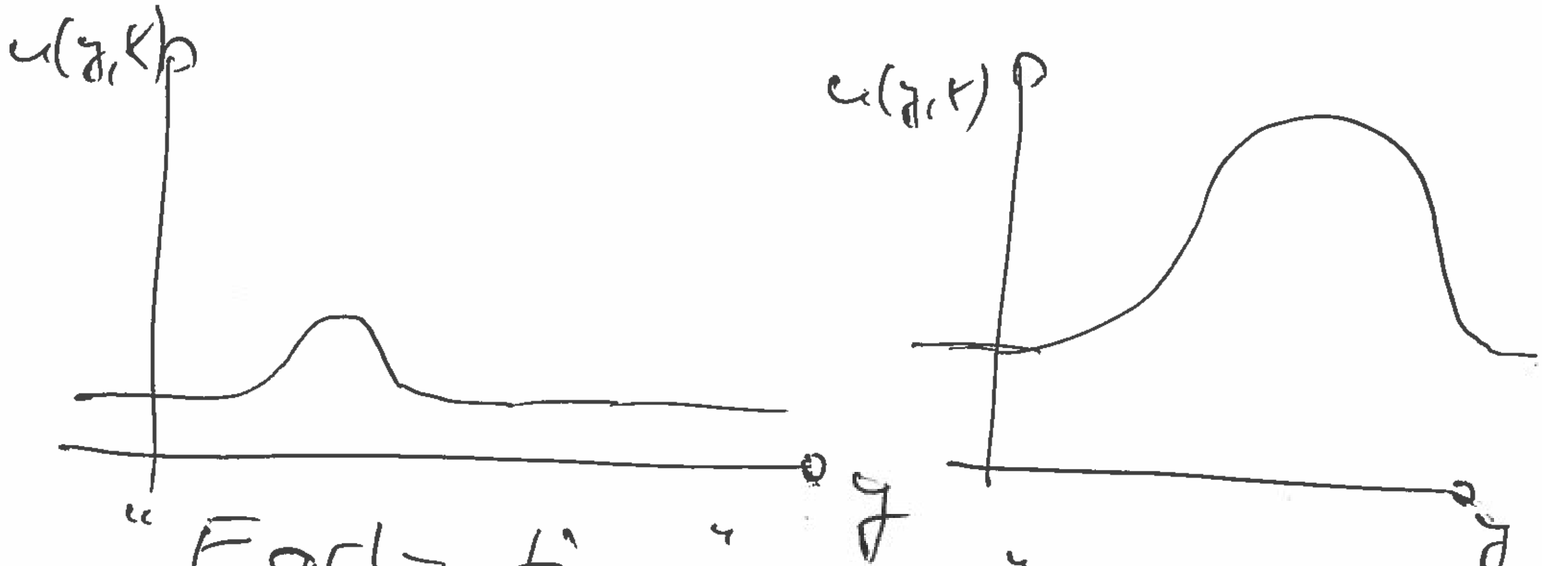
§.. Similarity solutions

Often solutions are self-similar; i.e. they have the same "shape" but possibly different scales at different times.

E.g: $u(y,t)$



"Early time" "Late time"

Such solutions have the form

$$u(y,t) = \underset{\uparrow \atop \text{amplitude}}{a(t)} \underbrace{f\left(\frac{y}{b(t)}\right)}_{\text{shape}}$$

Sim. solns. don't always exist – try it!

They often reduce PDEs into ODE for $f(\eta)$

where $\eta = \dfrac{y}{b(t)}$ ← similarity variable.

~~This~~ The existence of such solutions is often suggested by dimensional arguments.

Example:

Rayleigh's jerked plate.

$t > 0: \; U > 0$

Assume: parallel flow

$\underline{u} = u(y,t)\, \underline{e_x}$; $\nabla p = 0$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \text{(as before)}$$

<u>IC</u>: $t = 0$: $u = 0$

<u>BC</u>: $\left.\begin{array}{l} u(y=0,t) = U \quad (*) \\ u \to 0 \text{ as } y \to \infty \end{array}\right\}$ for $t > 0$

<u>Note</u>:
- $u = u(y, t; U, \nu)$
- PDE is linear & homog.
- BC/IC are linear in u & homogeneous apart from $(*)$

$\Rightarrow u(y, t; \nu, U)$ <u>must</u> be a linear fct. of U. (assuming uniqueness)

Proof: Assume have soln for U

$$2\frac{\partial u}{\partial t} = 2\nu \frac{\partial^2 u}{\partial y^2}$$

$$t=0: \quad 2u=0$$

$$\left.\begin{aligned} 2u(y=0,t) &= 2U \\ 2u &\to 0 \quad \text{as } y \to \infty \end{aligned}\right\} t>0$$

$\square$

$$u(y,t;U,\nu) = U f(y,t;\nu)$$

Now: Dimensions must match! Both sides must have dimensions m/sec.

$\Rightarrow$ f must be a non-dim. fct of its arguments.

$f(y, t; \nu)$:

$$\boxed{\begin{aligned}[y] &= m \\ [t] &= sec \\ [\nu] &= \frac{m^2}{sec}\end{aligned}}$$

$$[\nu] = \frac{\left[\frac{\partial u}{\partial t}\right]}{\left[\frac{\partial^2 u}{\partial y^2}\right]} = \frac{\left(\cancel{\frac{m}{sec^2}}\right)}{\left(\cancel{\frac{m}{sec\, m^2}}\right)}$$

$$[\nu] = \frac{m^2}{sec}$$

$$\eta = \frac{y^2}{\nu t} \quad \text{or} \quad \frac{\nu t}{y^2}, \ldots$$

we choose

$$\eta = \frac{y}{\sqrt{\nu t}}$$

So the soln. <u>must</u> have the form

$$u(y,t;\nu,U) = U f(\eta)$$

where $\boxed{\eta = y(\nu t)^{-1/2}}$

Into PDE:

$$\frac{\partial u}{\partial t} = U \underbrace{\frac{df}{d\eta}}\,\underbrace{\frac{\partial \eta}{\partial t}}$$

$$\boxed{\frac{\partial u}{\partial t} = U f' \frac{y}{\sqrt{\nu}}\left(-\frac{1}{2}\right) t^{-3/2}}$$

$$\frac{\partial u}{\partial y} = U f' \frac{\partial \eta}{\partial y} = U f' (\nu t)^{-1/2}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = U f'' \frac{1}{\nu t}}$$

into PDE:

$$\underbrace{\cancel{U} f' \frac{y}{\sqrt{\nu}}\left(-\frac{1}{2}\right) t^{-3/2}}_{\frac{\partial u}{\partial t}} = \cancel{\nu}\,\underbrace{\cancel{U} f'' \frac{1}{\cancel{\nu t}}}_{\frac{\partial^2 u}{\partial y^2}}$$

$$-\frac{1}{2} \underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta} f' = f''$$

$$\boxed{f'' + \frac{1}{2}\eta f' = 0}$$

ODE for $f(\eta)$.

$$\boxed{\begin{array}{l} u(y,t) = U f(\eta) \\ \eta = \frac{y}{\sqrt{\nu t}} \end{array}}$$

BC: $u = U$ at $y = 0$

$$\boxed{f(\eta = 0) = 1}$$

$u \to 0$ as $y \to \infty$

$$\boxed{f \to 0 \quad \text{as} \quad \eta \to \infty}$$

IC: $u \to 0$ as $t \to 0$

$$\boxed{f \to 0 \text{ as } \eta \to \infty}$$

again, phew!