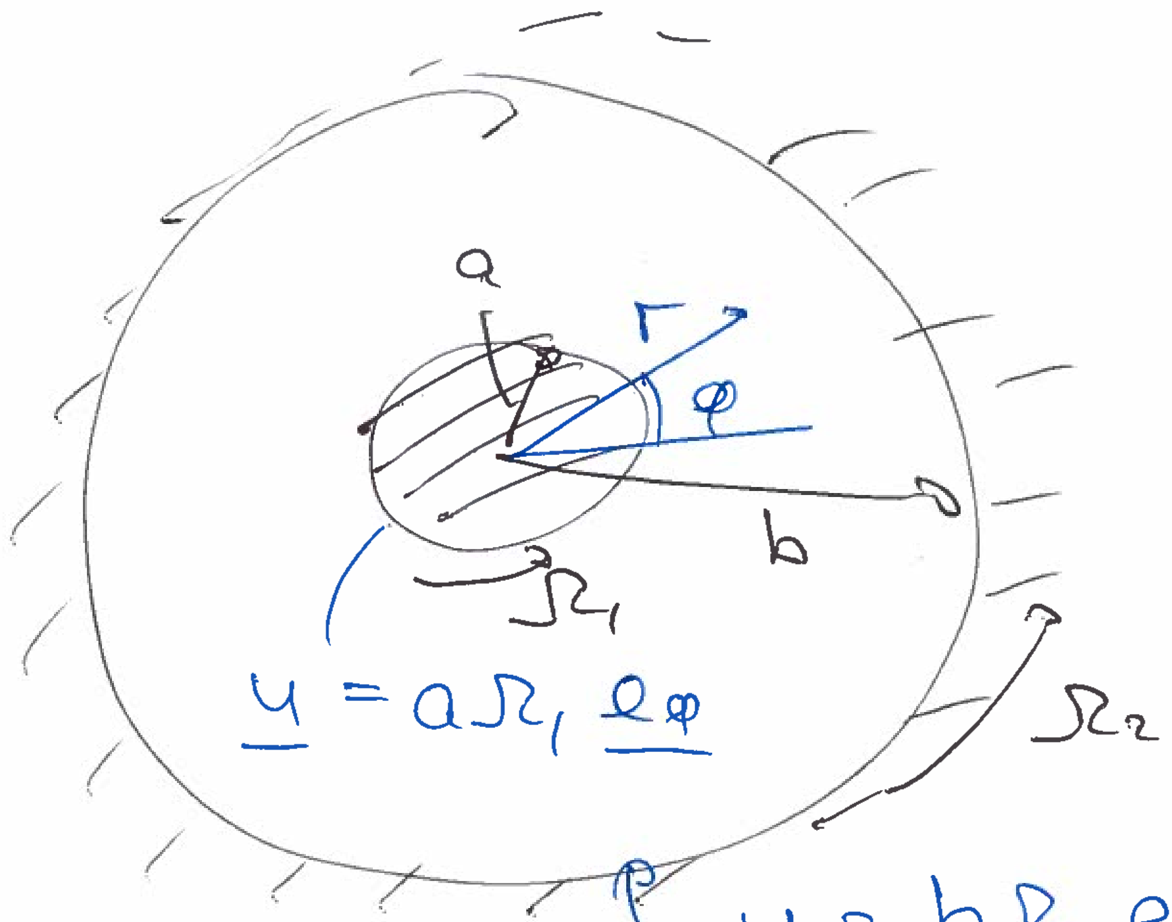
Example: Circular Couette flow



$\underline{u} = a\Omega_1 \underline{e_\varphi}$

$\underline{u} = b\Omega_2 \underline{e_\varphi}$

Assumptions:

- steady
- $\underline{u} = u_\varphi \underline{e_\varphi}$
- $\frac{\partial}{\partial\varphi} = 0$
- $\frac{\partial}{\partial z} = 0$

$$\left.\right\} \quad \underline{u} = \underbrace{u_\varphi(r)}_{v(r)} \underline{e_\varphi}$$

- Flow driven by wall:

$\nabla p = \underline{0}$.

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0

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \frac{v^2}{r} = -\cancel{\frac{1}{\rho}\frac{\partial P}{\partial r}} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = -\cancel{\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \cancel{\frac{2}{r^2}\frac{\partial u}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = -\cancel{\frac{1}{\rho}\frac{\partial P}{\partial z}} + \cancel{\nu\nabla^2 w},$$

✓

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

✓

$u = w = 0 \qquad v(r)$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

φ – comp. of N.St:

$$0 = \nabla^2 u - \frac{u}{r^2}$$

$$0 = \underbrace{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}}_{\nabla^2 u(r)} - \frac{u}{r^2}$$

$$0 = r^2 \frac{d^2 u}{dr^2} + r\frac{du}{dr} - u = 0$$

Euler ODE

Ansatz: $u \sim r^{\lambda}$

$$0 = r^2 \lambda(\lambda-1) r^{\lambda-2} + r\lambda r^{\lambda-1} - r^{\lambda} = 0$$

$$0 = r^{\lambda}\underbrace{\left(\lambda(\lambda-1)+\lambda-1\right)}_{=0} = 0$$

$$\lambda^2 = 1 \Rightarrow \lambda_1 = 1$$
$$\lambda_2 = -1$$

$$u(r) = Ar + B\frac{1}{r}$$

2 constants A & B from BC

$$u(r=a) = \Omega_1 a$$

$$u(r=b) = \Omega_2 b$$

⋮

$$u(r) = \frac{1}{b^2-a^2}\left\{(b^2\Omega_2 - a^2\Omega_1)\, r - \frac{a^2 b^2(\Omega_2-\Omega_1)}{r}\right\}$$

<u>Check:</u> BC ✓

$$\Omega_1 = \Omega_2 = \Omega$$

$\Rightarrow$ $u = \Omega r$ (rigid body rotation) ✓

But radial comp. of NSt is not satisfied!

$$+\frac{u^2}{r} = \cancel{\frac{1}{\rho}\frac{dp}{dr}}$$

So the assumption that
$\nabla p = \underline{0}$ is inconsistent
(centrifugal effects).

$\Rightarrow$ must allow

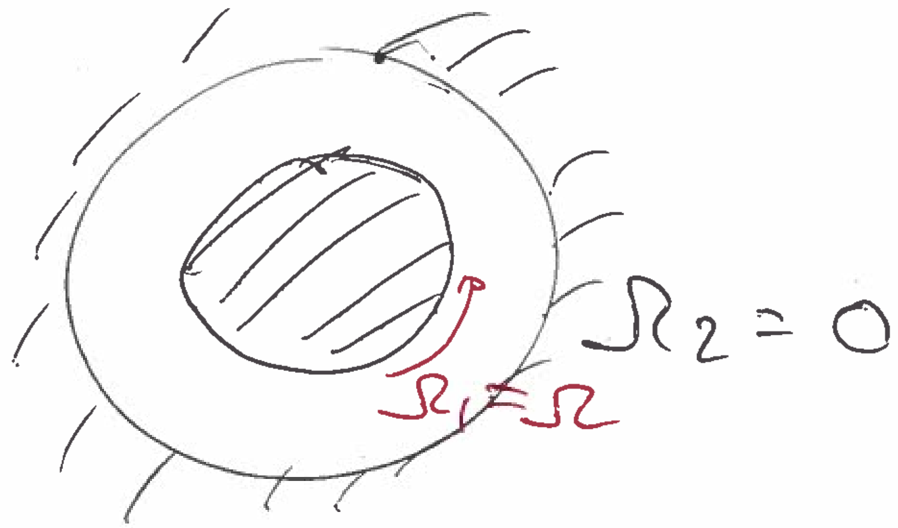
$$\nabla p = \frac{\partial p}{\partial r}\underline{e_r} \; ; \; p(r)$$

In that case:

$$\frac{dp}{dr} = +\rho \frac{u^2(r)}{r} \longrightarrow \text{given.}$$

(integrate).

Torque on inner cylinder



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$u = 0 \qquad v(r) \qquad w = 0$

$$\epsilon_{rr} = \cancel{\frac{\partial u}{\partial r}} \qquad \epsilon_{\varphi\varphi} = \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{u}{r}}$$

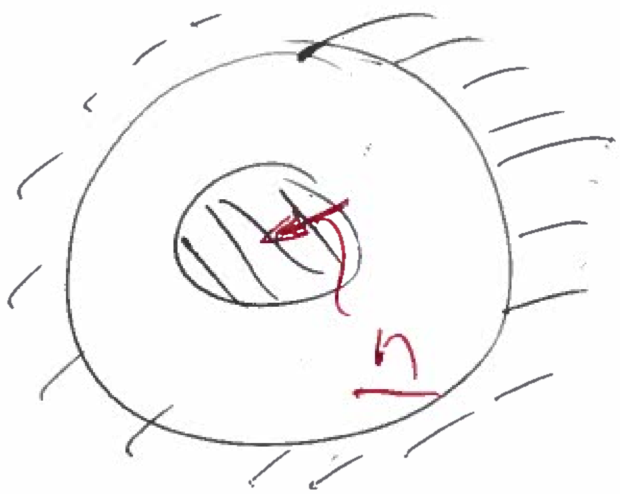
$$\epsilon_{zz} = \cancel{\frac{\partial w}{\partial z}} \qquad \boxed{\epsilon_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right.} + \left.\cancel{\frac{1}{r}\frac{\partial u}{\partial \varphi}}\right]$$

$$\epsilon_{\varphi z} = \frac{1}{2}\left[\cancel{\frac{1}{r}\frac{\partial w}{\partial \varphi}} + \cancel{\frac{\partial v}{\partial z}}\right] \qquad \epsilon_{rz} = \frac{1}{2}\left[\cancel{\frac{\partial u}{\partial z}} + \cancel{\frac{\partial w}{\partial r}}\right]$$

in this case:

$$v(r) = \frac{a^2 \Omega}{b^2 - a^2}\left(\frac{b^2}{r} - r\right)$$

Traction acting onto fluid @ $r = a$.



$n_r = -1$

$n_\varphi = 0$

$$\underline{n} = \underbrace{n_r}_{-1} \underline{e_r} + \underbrace{n_\varphi}_{0} \underline{e_\varphi}$$

$t_i = \tau_{ij} n_j \qquad (i,j) = (r, \varphi)$

$\tau_{ij} = -p\,\delta_{ij} + 2\mu \underbrace{e_{ij}}_{\text{from formula sheet.}}$

$e_{ij} = 0$ apart from

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$$e_{r\varphi} = \frac{1}{2} r \frac{\partial}{\partial r}\left(\frac{u(r)}{r}\right)$$

given

$$e_{r\varphi} = -\frac{a^2 b^2 \Omega}{b^2 - a^2} \frac{1}{r^2}$$

want this at $r = a$:

$$e_{r\varphi}\Big|_{r=a} = -\frac{b^2 \Omega}{b^2 - a^2}$$

$$t_i = -p n_i + 2\mu e_{ij} n_j$$

$i = $ "r":

$$t_r = -p \underbrace{n_r}_{-1} + 2\mu \left[\cancel{e_{rr}} n_r + e_{r\varphi} \cancel{n_\varphi} + e_{rz} \cancel{n_z}\right]$$

$$\underline{t_r = p(r=a)}$$

✓

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t_φ

$i = \varphi$:

$$t_\varphi = -p\cancel{n_\varphi} + 2\mu\left[e_{\varphi r} n_r + \cancel{e_{\varphi\varphi} n_\varphi} + \cancel{e_{\varphi z} n_z}\right]$$

$$t_\varphi = -2\mu e_{r\varphi}$$

$$t_\varphi = 2\mu \frac{b^2 \Omega}{b^2 - a^2}$$