§... Curvilinear coordinates

So far: Cartesian coordinates

$\underline{r} = (x, y, z) = (x_1, x_2, x_3)$

$\underline{u} = u_1 \underbrace{\underline{e_1}}_{\underline{e_x} = \underline{i}} + u_2 \underline{e_2} + u_3 \underline{e_3}$

index notation $\Big\}$ u_i

Eqns can be transformed to different coord. systems:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$ for scalar $\phi(x, y)$

$x = r \cos\varphi$

$y = r \sin\varphi$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2}$$

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N. St. eqns contain vectors; must transform these & their derivatives too.

$$\underline{u} = u_1 \underline{e_x} + u_2 \underline{e_2} + u_3 \underline{e_3}$$

$$= u_r \underline{e_r} + u_\varphi \underline{e_\varphi} + u_z \underline{e_z}$$

but here the basis vectors depend on the coords too!

<u>E.g.:</u>

$$\underline{e_r} = \cos\varphi\, \underline{e_x} + \sin\varphi\, \underline{e_y}$$

$\Longrightarrow$ Differential operators also act on the basis vectors!

$\Longrightarrow$ A MESS!

But we can still use index notation for certain eqns:

$$t_i = \tau_{ij} n_j \qquad (i,j) = (r, \varphi, z)$$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu\, e_{ij}$$

To illustrate the origin of some of the terms in the N. St. eqns in cyl. polars, consider flow assoc. with rigid body rotation.

$$\underline{u} = \underbrace{u_\varphi}_{\Omega r}\, \underline{e_\varphi} \; ; \quad u_r = u_z = 0$$

const.

$$\underline{u} = \underbrace{u_r}_{u=0} \, \underline{e_r} + \underbrace{u_\varphi}_{v=\Omega r} \, \underline{e_\varphi} + \underbrace{u_z}_{w=0} \, \underline{e_z}$$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} \boxed{- \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right],$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0.$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

r- comp of mom. eqn gives:

$$+\frac{v^2}{r} = +\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\rho\frac{\Omega^2 r^{\cancel{2}}}{\cancel{r}} = \frac{\partial p}{\partial r}$$

$$\frac{\partial p}{\partial r} = \rho\Omega^2 r$$

$$p(r) = p_0 + \frac{1}{2}\rho\Omega^2 r^2$$

This is a centrifugal effect.

Example:

Hagen-Poiseuille flow in pipe

driven by press. drop.

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Assume:

$$\underline{u} = u_z \underline{e_z} = \underbrace{u_z(r)}_{w(r)} \underline{e_z}$$

into r-comp: (check)

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \Rightarrow$$

φ-component:

$$0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \varphi}$$

press does not depend on r or φ

continuity

$$0 = 0 \quad ✓$$

z-comp: (check!)

$$0 = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{fct of } z \text{ only.}} + \underbrace{\nu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)}_{\text{fct of } r \text{ only.}}$$

$\frac{\mu}{\rho}$

As before (parallel flow) $\frac{\partial p}{\partial z} = G = \text{const}.$

$$\frac{G}{\mu} = \underbrace{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)}$$

$$\frac{G}{\mu} r = \frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$$

$$\frac{1}{2}\frac{G}{\mu}r^2 + A = r\frac{\partial w}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{1}{2}\frac{G}{\mu}r + \frac{A}{r}$$

$$w(r) = \frac{1}{4}\frac{G}{\mu}r^2 + A\ln r + B$$

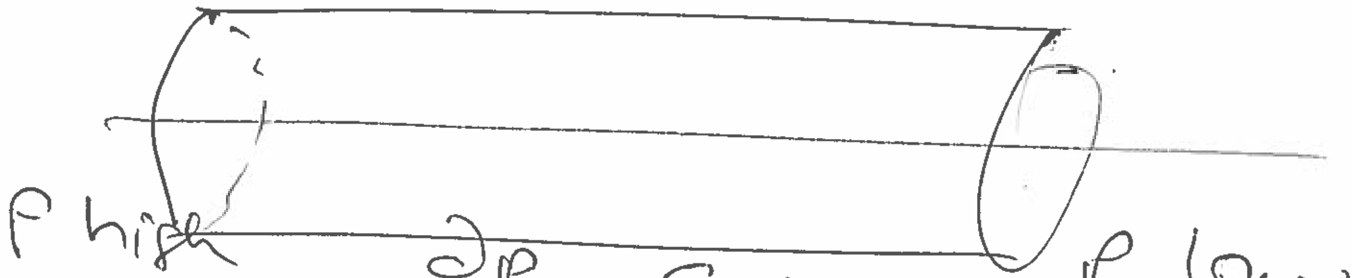
const. from BC.

No slip: $w(r=R) = 0$

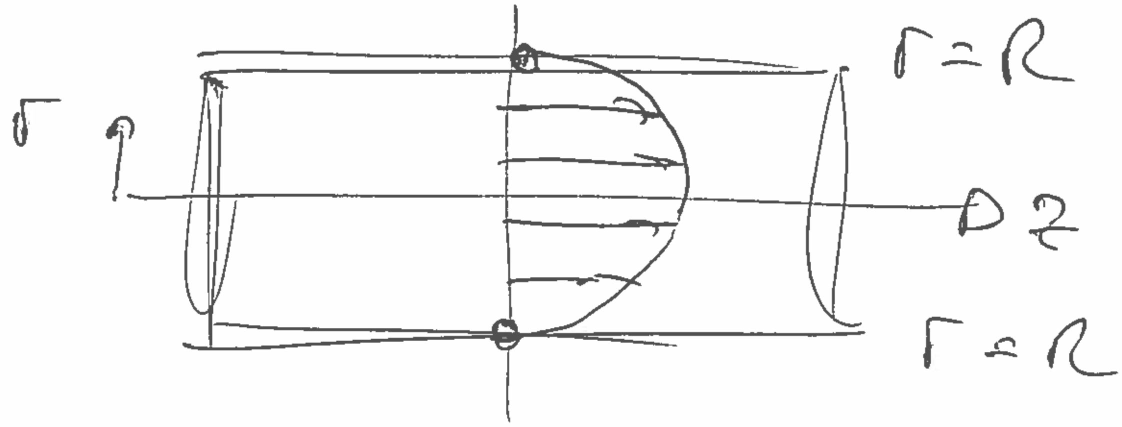
& soln is finite @ $r=0$

$\Rightarrow A = 0$

$$w(r) = \frac{1}{4}\frac{G}{\mu}\underbrace{\left(r^2 - R^2\right)}_{\le 0}$$



$$\frac{\partial p}{\partial z} = G < 0$$



$$w \geq 0$$