$\boxed{\underline{u} = u(\cancel{x}, y, z, t)\, \underline{e_x}}$ **Assumption!**

$$\rho \frac{\partial u}{\partial t} = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho f_y = \frac{\partial p}{\partial y}$$

$$\rho f_z = \frac{\partial p}{\partial z}$$

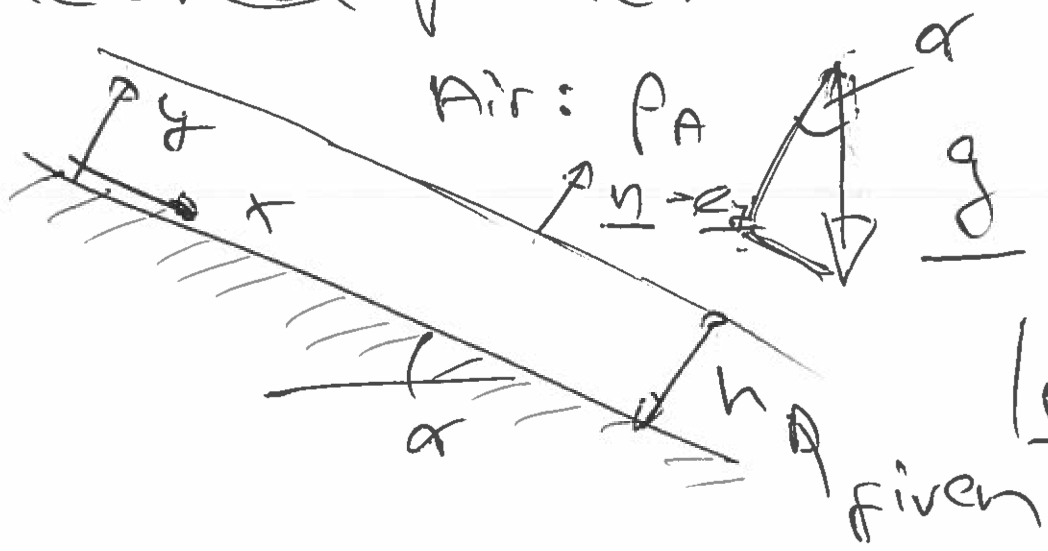
$F = 0$:

$$\frac{\partial u}{\partial t} = -\frac{G}{\rho} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial p}{\partial x} = G(t)$$

$$\nu = \frac{\mu}{\rho}$$

Example: Flow down an inclined plane:



$|\underline{g}| = g$

Decompose gravity into coordinate directions:

$$\underline{F} = \underline{g} = \underbrace{g \sin\alpha}_{F_x} \underline{e}_x - \underbrace{g \cos\alpha}_{F_y} \underline{e}_y$$

Assume: parallel, steady flow indep. of z.

$$\cancel{\rho \frac{\partial u}{\partial t}} = -\frac{\partial p}{\partial x} + \underbrace{\rho g \sin\alpha}_{F_x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}} \right)$$

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha} \quad (1)$$

y-comp:

$$\boxed{0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha = 0} \quad (2)$$

3

z-comp:

$$0 = -\frac{\partial p}{\partial z} \Rightarrow p = p(x,y)$$

BC: No slip at $y=0$

$$\boxed{u(y=0) = 0}$$

At $y = h = \text{const}$ (no need for kinematic BC.): viscous fluid meets the inviscid air at pressure p_A (given)

Air exerts a traction

$$\underline{t} = -p_A \underline{n} = -p_A \underline{e_y}$$

onto fluid.

$$t_i = \tau_{ij} n_j$$

$i=1$ x

2 y

3 z

at $y=h$

(4)

$t_1 = t_x = 0$ | $n_1 = n_x = 0$

$t_2 = t_y = -p_A$ | $n_2 = n_y = 1$

$t_3 = t_z = 0$ | $n_3 = n_z = 0$

$$t_i = \tau_{ij} n_j$$

$$\tau_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$t_i = -p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

sum over j: only $j=2$ contrib.

<u>$i=2$:</u>

$u_2 = 0$

$$t_2 = -p_A = -p n_2 + \mu\left(\cancel{\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}}\right) n_2$$

(all other terms vanish because $n_1 = n_3 = 0$)

$$\boxed{p(y=h) = p_A}$$

(5)

$i=1$:

$$t_1 = 0 = -\cancel{p n_1} + \mu\left(\frac{\partial u_1}{\partial x_2} + \cancel{\frac{\partial u_2}{\partial x_1}}\right) n_2$$

0 $\quad u_2 = 0 \quad$ 1

again only contributes $j=2$

$$\boxed{0 = \mu \frac{\partial u}{\partial y} \quad \text{at } y = h}$$

zero (tangential) shear stress.

$i=3$:

$$0 = 0$$

Integrate (2) w.r.t. y

$$\frac{\partial p}{\partial y} = -\rho g \cos\alpha$$

$$p(x,y) = -\rho g \cos\alpha \, y + f(x)$$

$i=2$ BC:

$$p_A = p(y=h) \quad \forall x \Rightarrow f(x) \text{ is constant.}$$

(6

$$\boxed{p(x,y) = p_A + \rho g \cos\alpha \,(h-y)}$$

pressure increases through film thickness. (hydrostatics)

Into (1):

$$0 = -\cancel{\frac{\partial p}{\partial x}} + \rho g \sin\alpha + \mu \frac{\partial^2 u}{\partial y^2}$$

integrate twice:

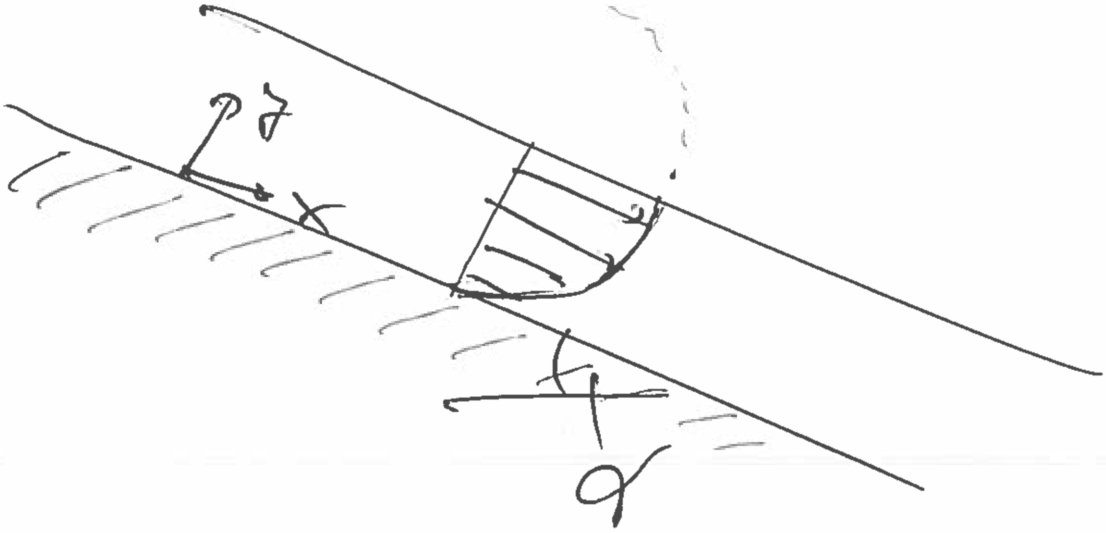
$$u = -\frac{1}{2} \frac{\rho g \sin\alpha}{\mu} y^2 + Ay + B$$

ν

BC: $u(y=0) = 0$

$$\mu \frac{\partial u}{\partial y}\bigg|_{y=h} = 0$$

$$\boxed{u = \frac{g \sin\alpha}{\nu}\left[hy - \frac{1}{2}y^2\right]}$$


z
x
α