Index notation

Various ways of expressing/writing a vector:

$$\underline{a} = a_1 \underset{\underline{i}}{\underline{e_1}} + a_2 \underset{\underline{j}}{\underline{e_2}} + a_3 \underset{\underline{k}}{\underline{e_3}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

↑ symbolic

in components relative to some basis

$$(\underline{e_1}, \underline{e_2}, \underline{e_3}) = (\underline{i}, \underline{j}, \underline{k})$$

Convention 1:

Simply write down one term of each vector eqn:

$$\underline{c} = \underline{a} + \underline{b} \implies c_i = a_i + b_i$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

→ i is a free index & takes values 1, 2, 3.

Example: $\nabla \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix} \Rightarrow \frac{\partial \phi}{\partial x_i}$

Convention 2: Summation convention. Rule: Automatically sum over repeated (dummy) indices:

Example:

$$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$= \cancel{\sum_{i=1}^{3}} u_i v_i = \underbrace{u_i v_i}$$

Note: Since the summation index does not feature in the result, its name is irrelevant

$$r = \underbrace{u_i v_i} = \underbrace{u_k v_k}$$

Example:

$$\operatorname{div} \underline{u} = \operatorname{div} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\nabla \cdot \underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \frac{\partial u_k}{\partial x_k} = \frac{\partial u_j}{\partial x_j}$$

Higher order "tensors"

So far: Have looked @ vectors & their components (= rank 1 tensors).

Higher-order tensors arise naturally in many contexts:

$$\underline{\sigma} = \underset{\sim}{T} \cdot \underline{n}$$

$\underline{\sigma}$: (stress) vector

$\underset{\sim}{T}$: stress tensor

$\underline{n}$: outer unit normal vector in index notation:

$$\sigma_i = T_{ij} \, n_j$$ (matrix vector product.)

<u>Note:</u> Not every object with two subscripts is a rank tensor!

One special second order tensor is the Kronecker Delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$[\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplication by δ_{ij} has an interesting property: (5

$$b_j = \underbrace{a_i \delta_{ij}}_{} = \sum_{i=1}^{3} a_i \delta_{ij} = a_j$$

δ_{ij} exchanges indices.