

# EXAMPLE SHEET IX

1

1) Scales:

$$u = U \tilde{u}$$

$$x = L \tilde{x}$$

$$v = V \tilde{v} = \frac{h_0}{L} U \tilde{v}$$

$$y = h_0 \tilde{y}$$

$$p = P \tilde{p} = \frac{\mu U}{h_0} \frac{1}{(h_0/L)} \tilde{p}$$

$$t = \frac{L}{U} \tilde{t}$$

[remember: The scale for  $v$  was derived from the requirement that the terms in the continuity equation balance each other; the pressure scale followed from the requirement that the pressure gradient must balance the viscous dissipation]

into  $y$ -N.S.E.

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$$\left( \frac{\left( \frac{h_0}{L} \psi \right)}{\left( \frac{L}{h_0} \right)} \frac{\partial \tilde{\psi}}{\partial \tilde{t}} + \psi \frac{\left( \frac{h_0}{L} \psi \right)}{L} \frac{\partial}{\partial x^2} + \frac{h_0}{L} \psi \frac{\left( \frac{L}{h_0} \psi \right)}{h_0} \frac{\partial}{\partial y^2} \right)$$

$$= - \frac{\left( \frac{\mu \psi}{h_0} \right)}{\left( \frac{h_0}{L} \right) h_0} \frac{\partial p}{\partial r} + \mu \left( \frac{\left( \frac{h_0}{L} \psi \right)}{L^2} \frac{\partial^2}{\partial x^2} + \frac{\left( \frac{L}{h_0} \psi \right)}{h_0^2} \frac{\partial^2}{\partial y^2} \right)$$

$$\frac{g h_0 \psi^2}{L^2} \frac{D \tilde{\psi}}{D \tilde{t}} = - \frac{\mu \psi L}{h_0^3} \frac{\partial p}{\partial r} + \frac{\mu \psi}{h_0 L} \left( \left( \frac{h_0}{L} \right)^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

$$\underbrace{\frac{g \psi h_0}{\mu} \left( \frac{h_0}{L} \right)}_{Re} \frac{D \tilde{\psi}}{D \tilde{t}} = - \left( \frac{L}{h_0} \right)^2 \frac{\partial p}{\partial r} + \left( \frac{h_0}{L} \right)^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$\frac{h_0}{L} = \epsilon \ll 1$ ; Also assume (as before) that  $\epsilon Re \ll 1$ .

$$Re \epsilon \frac{D \tilde{\psi}}{D \tilde{t}} = - \left( \frac{1}{\epsilon} \right)^2 \frac{\partial p}{\partial r} + \epsilon^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

This term is  $O\left(\frac{1}{\epsilon^2}\right) \gg 1$  (order then all other terms).

There are no undetermined scales left, so to leading order (i.e., neglecting terms of size  $Re\epsilon$  and  $\epsilon^2$  against terms of size "1") the Navier-Stokes eqns can be approximated by

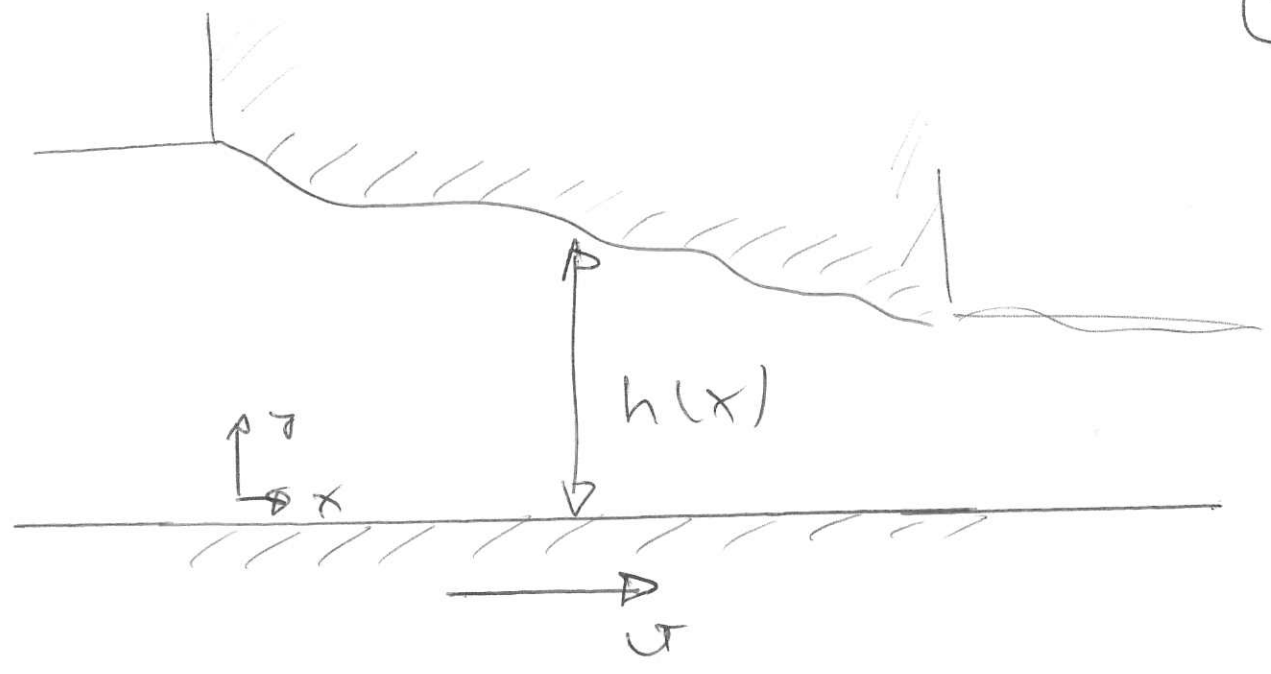
$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

&

$$\frac{\partial p}{\partial y} = 0$$

In other words, the wall slope changes so slowly that locally the flow behaves as if it were a parallel flow through a channel of constant width.

2)



$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

$$u(y=0) = u$$

$$u(y=h(x)) = 0$$

Since  $\frac{\partial p}{\partial y} = 0$  we can integrate w.r.t  $y$ :  
 & apply the BC:

$$u(x,y) = -\frac{1}{2\mu} \frac{\partial p}{\partial x} y (h(x) - y) + u \left(1 - \frac{y}{h(x)}\right)$$

Additional requirement: The same volume flux has to pass through every cross section.

$$Q = \int_0^{h(x)} u(x,y) dy = \text{const.} \quad (\text{and } -\text{os } x \text{ is unknown})$$

$$Q = -\frac{1}{2\mu} \frac{\partial p}{\partial x} \left( \frac{h^3}{2} - \frac{h^3}{3} \right) + \mu \left( h - \frac{h}{2} \right)$$

$$Q = -\frac{h^3}{12\mu} \frac{\partial p}{\partial x} + \frac{1}{2} \mu h$$

So:

$$\frac{\partial p}{\partial x} = \frac{12\mu}{h^3} \left( \frac{h}{2} \mu - Q \right)$$

integrate w.r.t.  $x$  & use  $p(x=0) = p_0$ :

$$p(x) = p_0 + 6\mu \int_0^x \left( \frac{\mu}{h^2(s)} - \frac{2Q}{h^3(s)} \right) ds$$

The second pressure B.C. provides an eqn. for  $Q$ :

$$p(L) = p_0 :$$

$$\int_0^L \left( \frac{\mu}{h^2(s)} - \frac{2Q}{h^3(s)} \right) ds = 0$$

$$Q = \frac{1}{2} \mu \frac{\int_0^L h^{-2}(s) ds}{\int_0^L h^{-3}(s) ds}$$

(ii) Now for  $h(x) = h_L + \frac{x}{L}(h_R - h_L)$  (6)

$$\int_0^L \left( h_L + \frac{x}{L}(h_R - h_L) \right)^{-N} dx =$$

$$= \frac{L}{h_R - h_L} \int_{h_L}^{h_R} z^{-N} dz = \frac{L}{h_R - h_L} \frac{1}{(-N+1)} \left( h_R^{-N+1} - h_L^{-N+1} \right)$$

So:

$$Q = \frac{1}{2} \psi \frac{\left(-\frac{1}{1}\right) \left(\frac{1}{h_R} - \frac{1}{h_L}\right)}{\left(-\frac{1}{2}\right) \left(\frac{1}{h_R^2} - \frac{1}{h_L^2}\right)} = \psi \frac{h_R^2 h_L^2 (h_L - h_R)}{(h_L^2 - h_R^2) h_R h_L}$$

$$\underline{\underline{Q = \psi \frac{h_R h_L}{h_L + h_R}}}$$

So:

$$\frac{p(x) - p_0}{6\mu U} = \int_0^x \left( h^{-2}(s) - 2 \frac{h_R h_L}{h_R + h_L} h^{-3}(s) \right) ds$$

Again:

$$\int_0^x \left( h_L + \frac{s}{L} (h_R - h_L) \right)^{-2} ds =$$

$$= \frac{L}{h_R - h_L} \int_{h_L}^{h(x)} z^{-2} dz = \frac{L}{h_R - h_L} \frac{1}{(-1+1)} \left( h^{-1}(x) - h_L^{-1} \right)$$

So:

$$\frac{p(x) - p_0}{6\mu U L} = \frac{1}{h_R - h_L} \left[ -\frac{1}{1} \left( \frac{1}{h} - \frac{1}{h_L} \right) - 2 \frac{h_R h_L}{h_R + h_L} \left( -\frac{1}{2} \right) \left( \frac{1}{h^2} - \frac{1}{h_L^2} \right) \right]$$

$$= \frac{1}{h_R - h_L} \left[ \frac{h(h - h_L)}{h^2 h_L} + \frac{h_R h_L}{h_R + h_L} \frac{h_L^2 - h^2}{h^2 h_L} \right]$$

$$= \frac{1}{(h_R^2 - h_L^2) h^2 h_L} \left[ (h^2 - h h_L)(h_R + h_L) + h_R h_L^2 - h_R h^2 \right]$$

$$= \frac{1}{(h_R - h_L) h^2} \left[ \cancel{h^2 h_R} + \cancel{h^2 h_L} - \cancel{h h_L h_R} - \cancel{h h_L^2} + \cancel{h_R h_L^2} - \cancel{h_R h^2} \right]$$

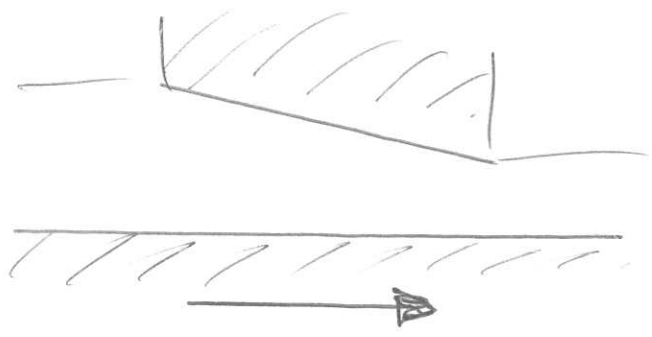
$$= \frac{1}{(h_R - h_L) h^2} \underbrace{(h^2 - h h_R - h h_L + h_R h_L)}_{(h_L - h)(h_R - h)}$$

$$\frac{p(x) - p_0}{6 \mu U L} = \frac{(h_R - h(x))(h_L - h(x))}{(h_R - h_L) h^2(x)}$$


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Now:  $h(x)$  lies between  $h_R$  &  $h_L$   
 so the numerator of this expression is neg.

Hence if  $h_R < h_L$ , i.e.:



$p(x) > 0$  under the bearing.



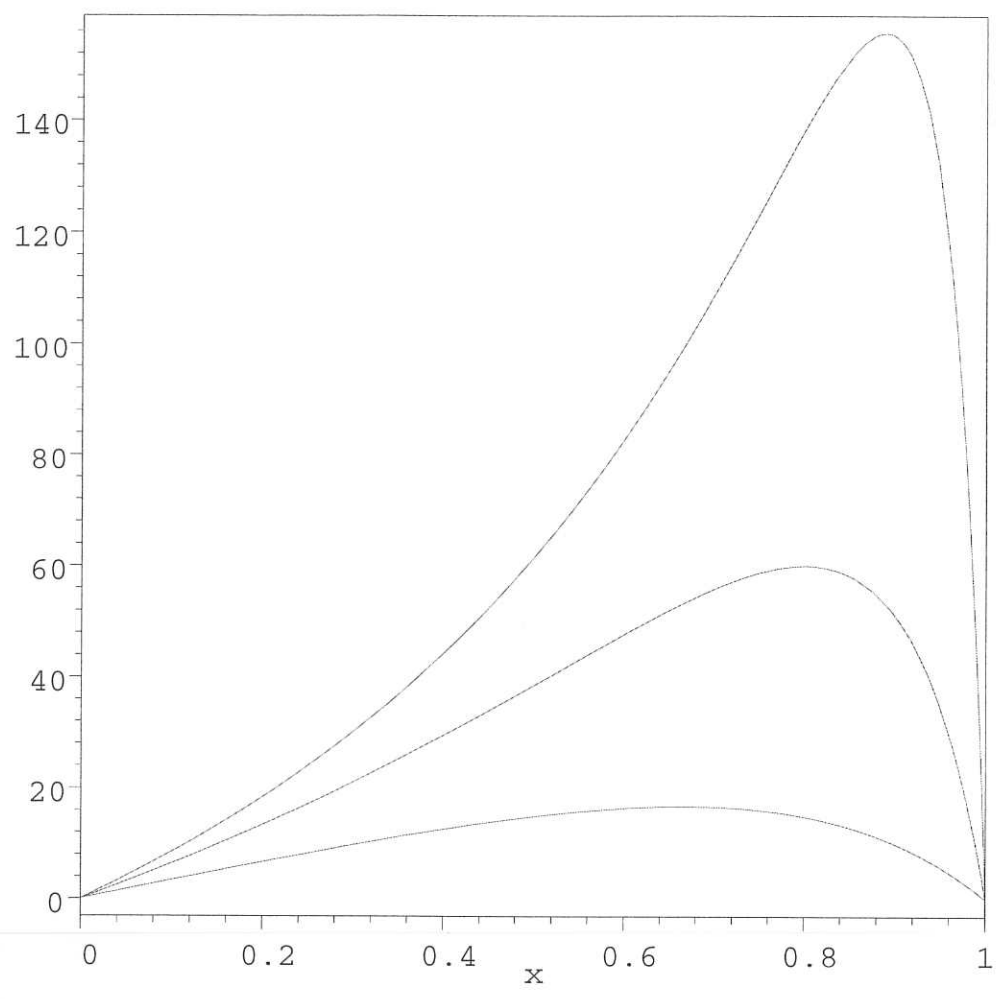
As the fluid is drawn into the narrowing lubrication gap the pressure exceeds the external pressure & produces a net upwards force, which keeps the surfaces from coming into contact.

Note that if the average height of the lubrication gap is  $h_0$  then:

$$\frac{\rho \mu U L}{6 \mu U L} \sim \frac{1}{h_0^2}$$

Hence, tremendous pressures can be generated in narrow gaps.

$$\frac{p(x) - p_0}{6\mu UL}$$



for  $L=1$ ,  $h_L=0.1$  &

$h_R = 0.05, 0.025$  &  $0.0125$

