Parallel flows

N.St. eqns are very complicated because of the nonlin. terms in $\frac{Du}{Dt}$.
These terms vanish in a variety of practically relevant flows.

Assume here that the flow is unidirectional.
w.l.o.g. assume

$$\boxed{\begin{aligned} &u_1 = u \text{ nonzero } \& \\ &u_2 = v = 0 \\ &u_3 = w = 0 \end{aligned}}$$

Note: u can still vary as a fct. of all coords & time.

$(x_1, x_2, x_3) = (x, y, t)$

$(u_1, u_2, u_3) = (u, v, w)$ 1

Kinematic viscosity $\nu = \frac{\mu}{\rho}$

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{w\frac{\partial v}{\partial z}} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{v\frac{\partial w}{\partial y}} + \cancel{w\frac{\partial w}{\partial z}} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\cancel{\frac{\partial^2 w}{\partial x^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}}\right)$$

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

$v = w = 0$

$u(\cancel{x}, y, z, t) \Rightarrow u(y, z, t)$

Separate sheet:

$$\boxed{\begin{aligned} &\rho \frac{\partial u}{\partial t} = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (1) \\ &\rho f_y = \frac{\partial p}{\partial y} \quad (2) \\ &\rho f_z = \frac{\partial p}{\partial z} \quad (3) \end{aligned}}$$

(parallel flow eqns)

$$\underline{F} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \text{given body force.}$$

$\Rightarrow$ linear eqns for $u(y,z,t)$ & $p(x,y,z,t)$

Special case: No body force

$\underline{F} = \underline{0}$

(2) & (3) $\Rightarrow$ $p = p(x,t)$

into (1): $u = u(y,z,t)$

$$\underbrace{\rho \frac{\partial u}{\partial t}}_{\text{fct of } (y,z,t)} = -\frac{\partial p}{\partial x} + \underbrace{\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\substack{\text{fct of } (y,z,t) \\ \text{not fct. of } x!}}$$

$\frac{\partial p}{\partial x}$ must also be indep. of x

$$\frac{\partial p}{\partial x} = \text{fct. of } (\cancel{y},\cancel{z},t) \text{ only}$$

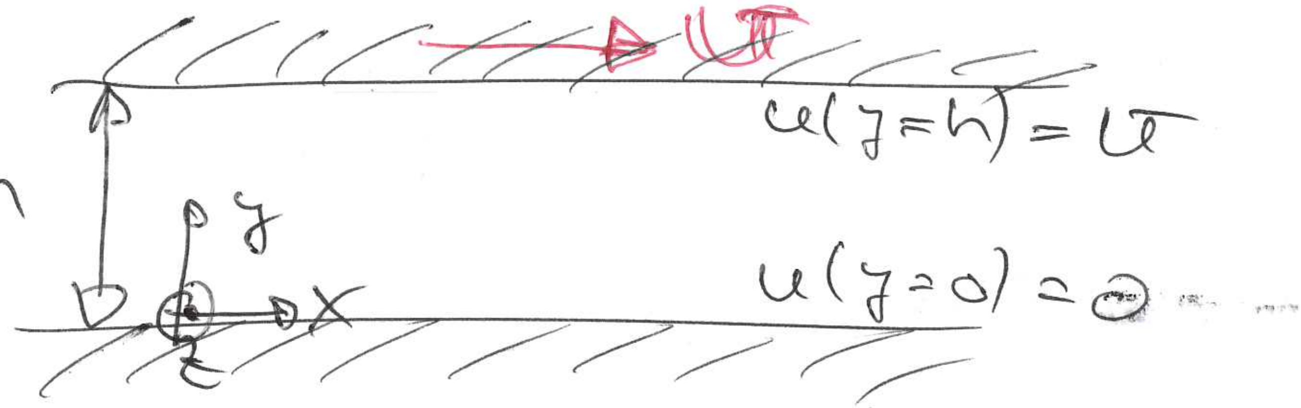
$$\frac{\partial p}{\partial x} = G(t)$$

<u>Parallel flow eqns without body force</u>

$$\boxed{\begin{aligned} &\rho \frac{\partial u}{\partial t} = -G(t) + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ &\frac{\partial p}{\partial x} = G(t) \\ &v = w = 0 \end{aligned}}$$

Example: Couette flow

Flow between parallel infinite plates; upper plate moves tangentially with veloc. U.



Assumptions:

- flow is unidirectional!
- — " — steady: $\frac{\partial}{\partial t} = 0$
- $\frac{\partial}{\partial z} = 0$
- $\frac{\partial p}{\partial x} = G = 0$

$$\cancel{\rho \frac{\partial u}{\partial t}} = -\cancel{G} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

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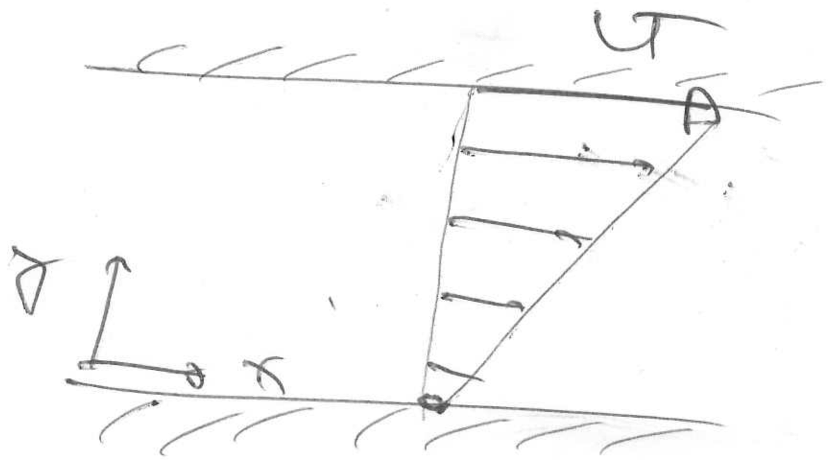
$$\frac{d^2u}{dy^2} = 0$$

2nd order ODE for $u(y)$

$$u(y) = Ay + B$$

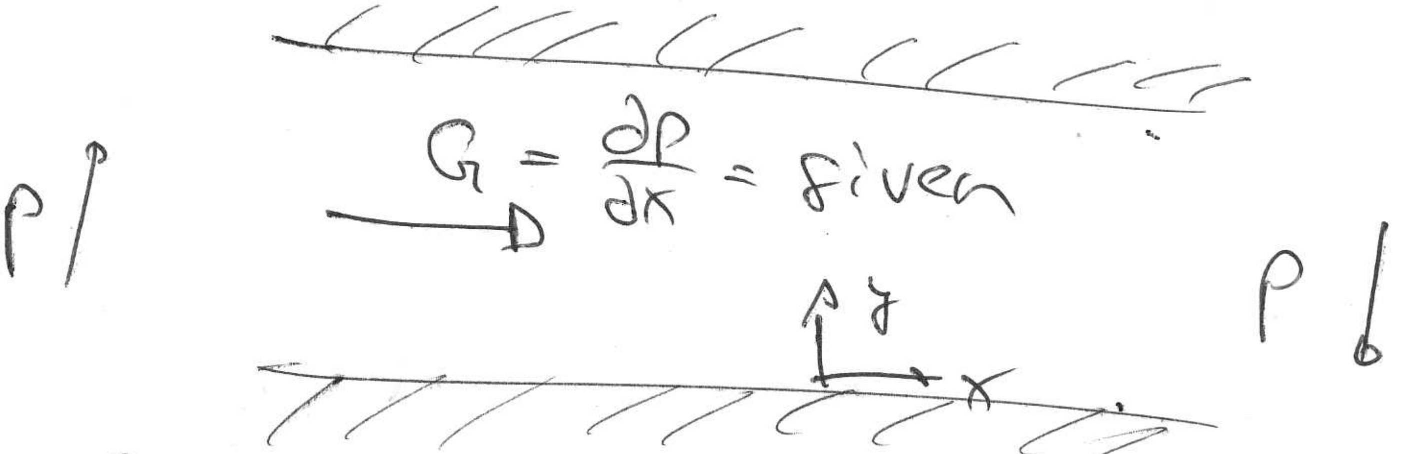
2 const. of integration from BC.

$$u(y) = \frac{U}{h}y$$



Example: Poiseuille flow

Pressure driven flow in a rigid channel.



Assume: As above but $G \neq 0$

$u(y)$

into eqns

$$\rho \cancel{\frac{\partial u}{\partial t}} = -G + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\boxed{\frac{d^2 u}{dy^2} = \frac{G}{\mu}}$$

2 BC:

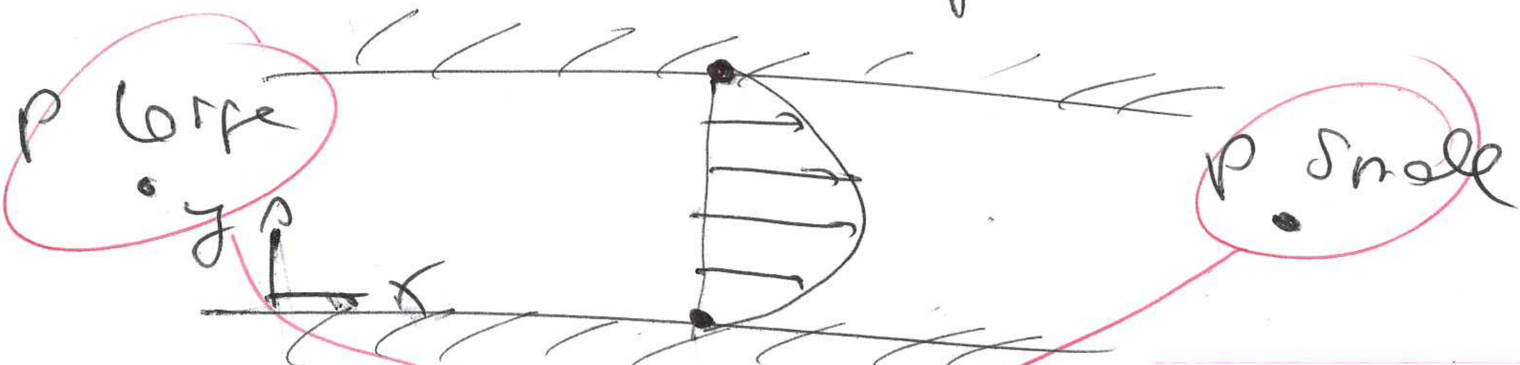
$u(y=0) = 0$

$u(y=h) = 0$

$$u(y) = \frac{1}{2}\frac{G}{\mu}y^2 + Ay + B$$

const. from B.C.

$$\boxed{u(y) = \frac{G}{2\mu}\underbrace{(y^2 - hy)}_{\text{neg.}}}$$



picture for $G < 0$

$$G = \frac{dp}{dx} < 0$$