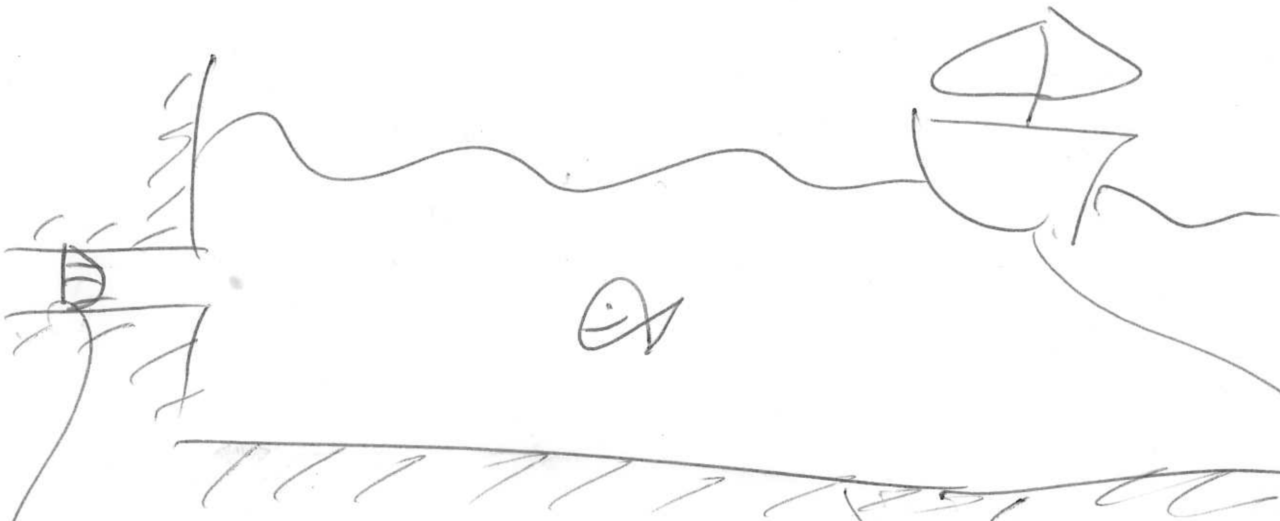
$$\boxed{\begin{aligned} \rho \frac{Du_i}{Dt} &= -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i \\ \frac{\partial u_j}{\partial x_j} &= 0 \end{aligned}}$$

momentum eqns.



(i)

$\underline{u} = \underline{u}_{inflow}$ (given)

(ii)

$\underline{u} = \underline{u}_{solid}$ (given)

(iii) Free surfaces

need 2 conditions:

- ~~kinem~~ kinematic BC
- traction BC

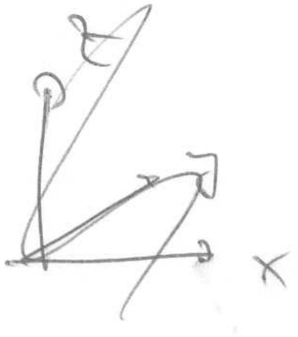
(a) Kinematic BC

The posn. of the free surface can always be described implicitly as

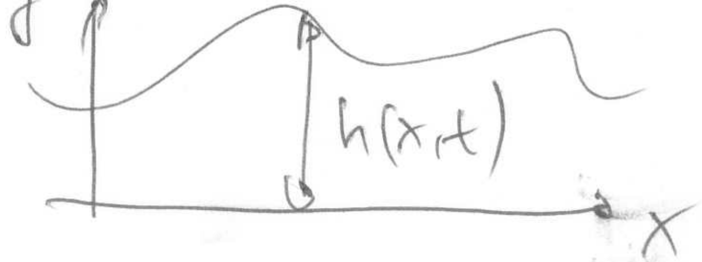
$$\boxed{f(x, y, z, t) = 0}$$

At least locally this can always be inverted to:

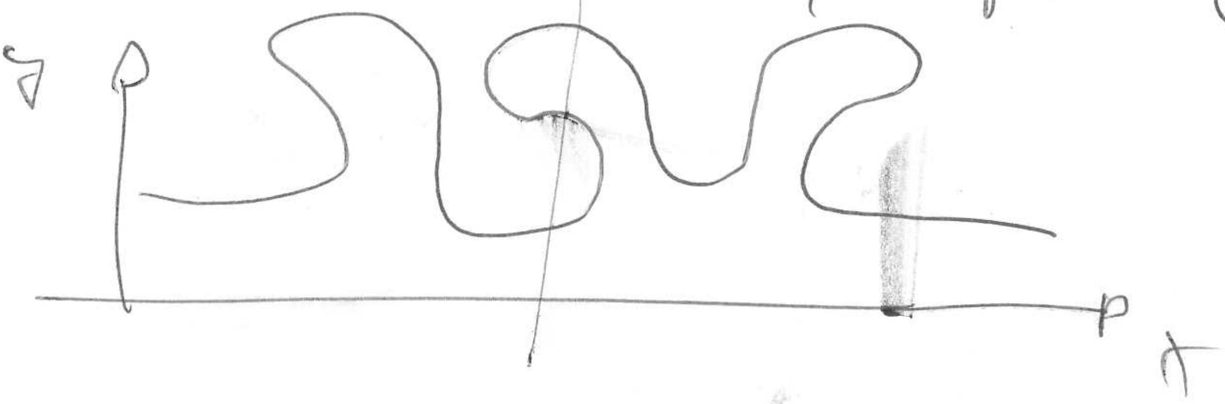
$$z = h(x, y, t)$$



2D $\quad y = h(x, t)$



won't work for, e.g. (3

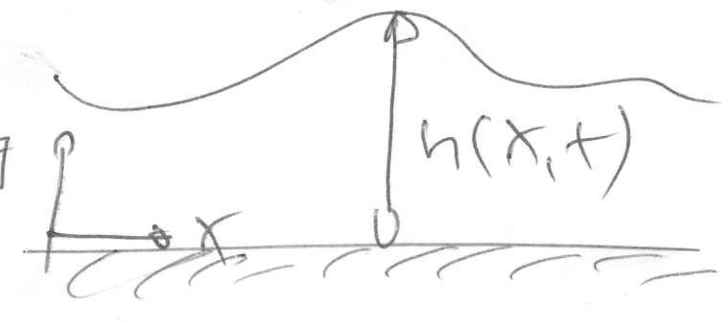


Physics: fluid particles on a free surface always stay on that free surface

$$\Rightarrow \boxed{\frac{DF}{Dt} = 0}$$

Example:

2D



$$y = h(x,t)$$

choose: $F(x,y,t) = h(x,t) - y = 0$

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \underbrace{u_j \frac{\partial F}{\partial x_j}}$$

$x_1 = x$
$x_2 = y$
$u_1 = u$
$u_2 = v$

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y}$$

$$\frac{\partial F}{\partial t} = \frac{\partial h}{\partial t}$$

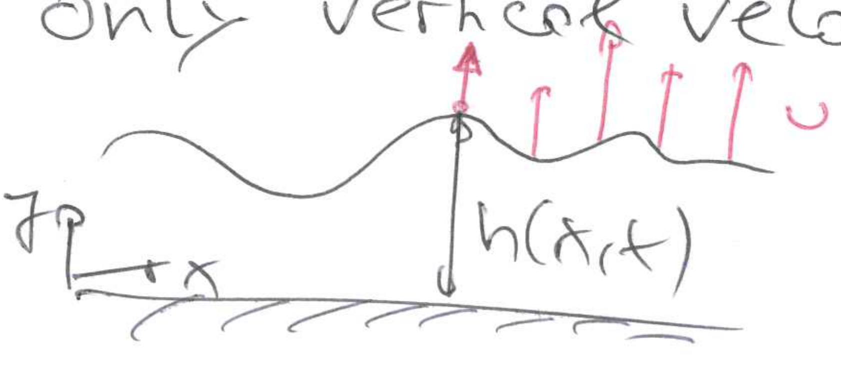
$$\frac{\partial F}{\partial x} = \frac{\partial h}{\partial x}$$

$$\frac{\partial F}{\partial y} = -1$$

$$\frac{DF}{Dt} = \boxed{\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - v = 0}$$

Special case 1 $\quad u = 0$

only vertical veloc.



$$\frac{\partial h}{\partial t} - v = 0$$

$$\frac{\partial h}{\partial t} = v$$

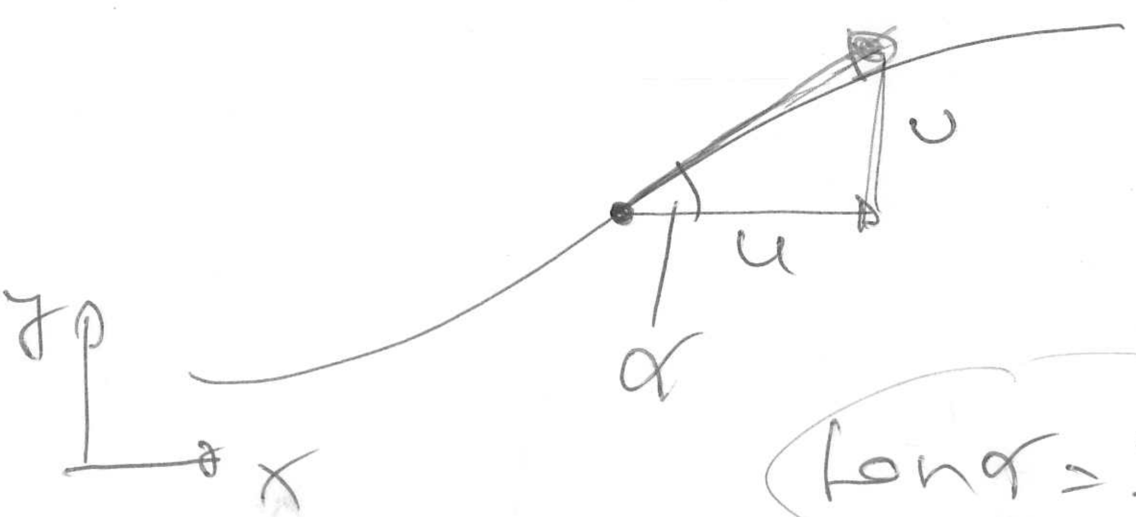
Special case 2

fixed free surface

$$\frac{\partial h}{\partial t} = 0$$

$$\cancel{\frac{\partial h}{\partial t}} + u\frac{\partial h}{\partial x} - v = 0$$

$$\frac{\partial h}{\partial x} = \frac{v}{u} = \frac{dh}{dx}$$



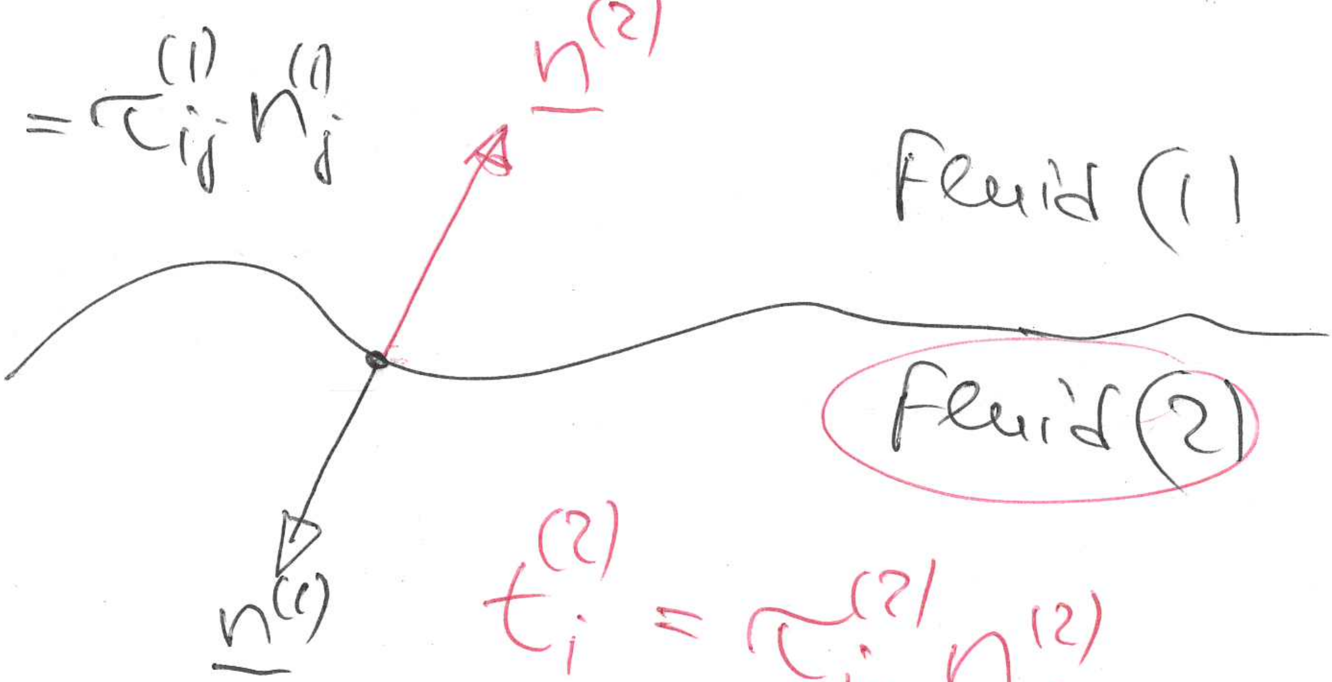
$$\tan\alpha = \frac{v}{u} = \frac{dh}{dx}$$

veloc. is tangential to surface

(b) traction BC

<u>Physics:</u> Stress must be continuous across the free surface (apart from surface tension effects).

$$t_i^{(1)} = \tau_{ij}^{(1)} n_j^{(1)}$$



$$t_i^{(2)} = \tau_{ij}^{(2)} n_j^{(2)}$$

$\underline{t}^{(1)}$ is traction acting <u>onto</u> fluid (1).

"Action = Reaction"

$$\underline{t}^{(1)} = -\underline{t}^{(2)}$$

$$\underline{n}^{(1)} = -\underline{n}^{(2)} \quad \text{(geometry)}$$

$$\boxed{\tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j}$$

where n_j is either one of the unit normals.
This is the traction BC.

Example: hydrostatics

$$\tau_{ij} = -p\,\delta_{ij}$$

$$\tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j$$

$$-p^{(1)}\delta_{ij} n_j = -p^{(2)}\delta_{ij} n_j$$

$$-p^{(1)} n_i = -p^{(2)} n_i$$

$$-p^{(1)} \underline{n} = -p^{(2)} \underline{n}$$

$$\underbrace{\left(p^{(1)} - p^{(2)}\right)}_{=0} \underline{n} = \underline{0}$$

$$p^{(1)} = p^{(2)}$$