Cauchy's eqn

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{\partial \tau_{ij}}{\partial x_j}+\rho f_i$$

$\Rightarrow$ rate of change of veloc!

but what is τ_{ij}?

$\Rightarrow$ Constitutive eqns

Observation:

fluids:
- can generate hydro-static pressures.
- have a resistance to shear (pull knife out of honey): viscosity.
- do not generate internal stresses when subjected to rigid body motions

This suggests that τ_{ij} should have a hydrostatic pressure contribution & depend on the rate of strain tensor ε_{ij}.

$$\tau_{ij} = \underbrace{-p\delta_{ij}}_{\text{hydrostatic pressure}} + \underbrace{2\mu\varepsilon_{ij}}_{\substack{\text{viscous stresses} \\ \text{depend linearly} \\ \text{on } \varepsilon_{ij}}} \qquad (*)$$

μ = (dynamic) viscosity, a property of the fluid that has to be determined experimentally

fluids that satisfy (*) are known as Newtonian fluids.

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$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Assume also: fluid is incompressible.. Into Cauchy's eqn:

$$\rho\frac{Du_i}{Dt} = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$= \rho f_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

$$= \rho f_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \cancel{\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)}$$

$\operatorname{div} \underline{u} = 0$

$$\boxed{\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2}}$$

& eqn. of continuity

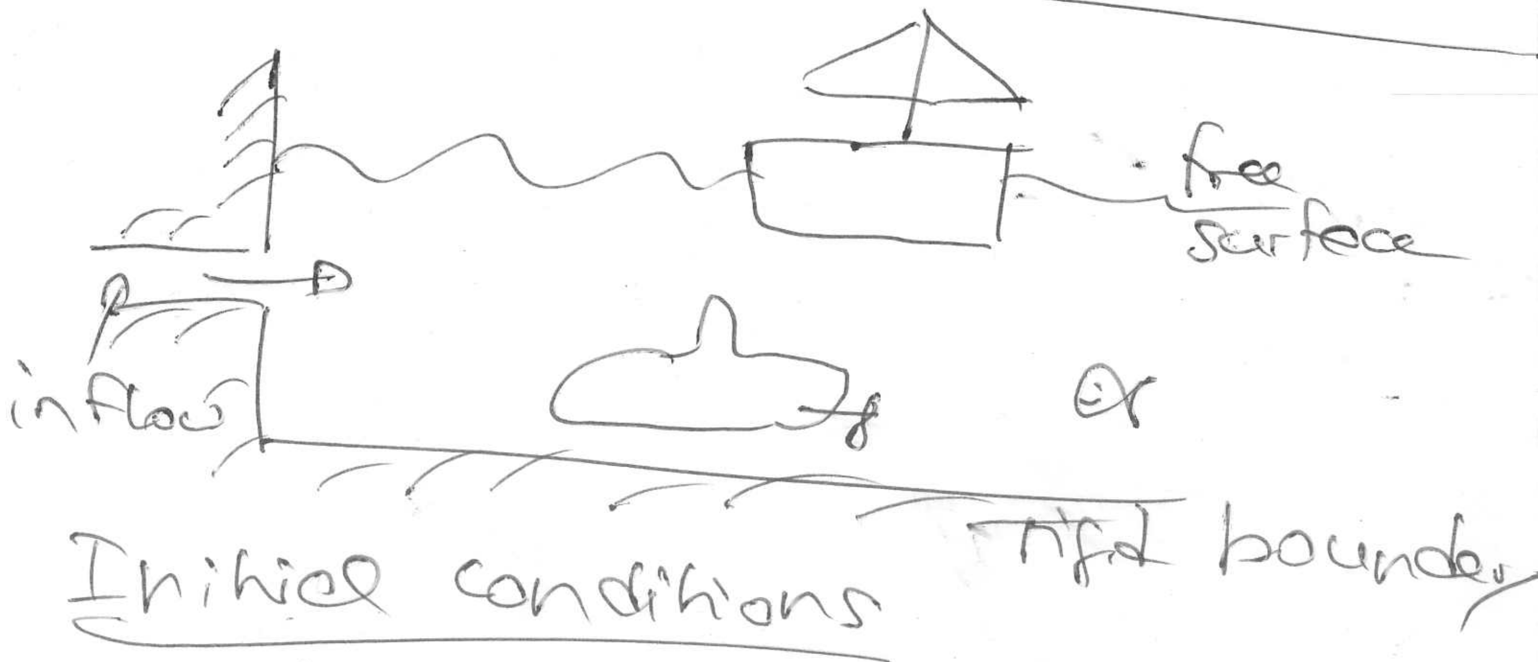
$$\boxed{\frac{\partial u_j}{\partial x_j} = 0}$$

These are the famous Navier Stokes eqns!

They are:

a system of four nonlinear, coupled PDEs of 2nd order in space & 1st order in time for three veloc components & pressure.

Boundary & initial conditions!



Initial conditions

Need to specify $u_i(x_j, t=0)$ for time dep. problems

Note: No IC for pressure.

Boundary conditions

(i) Inflow/outflow BC

$$\boxed{u_i = U_i}$$

given veloc. on the boundary

(ii) on solid surfaces

Observation: "No slip & no penetration"

solid veloc = fluid veloc

given

$$\boxed{u_i = U_i}$$

given veloc. of the solid.