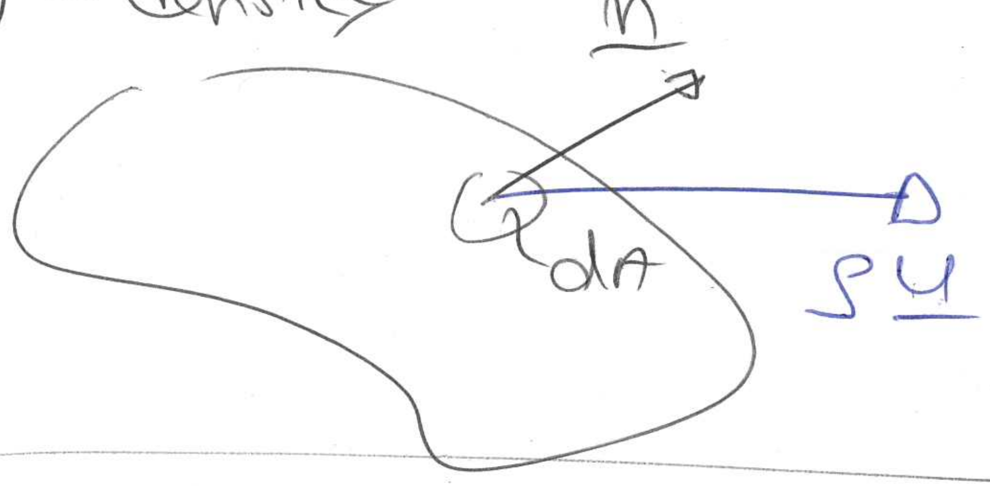
Continuity

$\rho(x_i, t)$ = density $\quad \frac{kg}{m^3}$



$$\boxed{-\oint \rho \underline{u} \cdot \underline{n} \, dA = \int \frac{\partial \rho}{\partial t} \, dV}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \underline{u}) = 0}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial \rho}{\partial x_j} u_j}_{u_j \frac{\partial \rho}{\partial x_j}} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

$$\boxed{\frac{D\rho}{Dt} + \rho \frac{\partial u_j}{\partial x_j} = 0}$$

Incompressible fluids ...

$$\frac{D\rho}{Dt} = 0$$

$$\boxed{\frac{\partial u_j}{\partial x_j} = \operatorname{div} \underline{u} = 0}$$

... have divergence free flow fields.

§(n+2) Stress, Cauchy's eqn & the Navier Stokes eqns

① The concept of stress/traction

Consider a finite blob of fluid loaded by some distributed force (pressure, shear stress, ...)

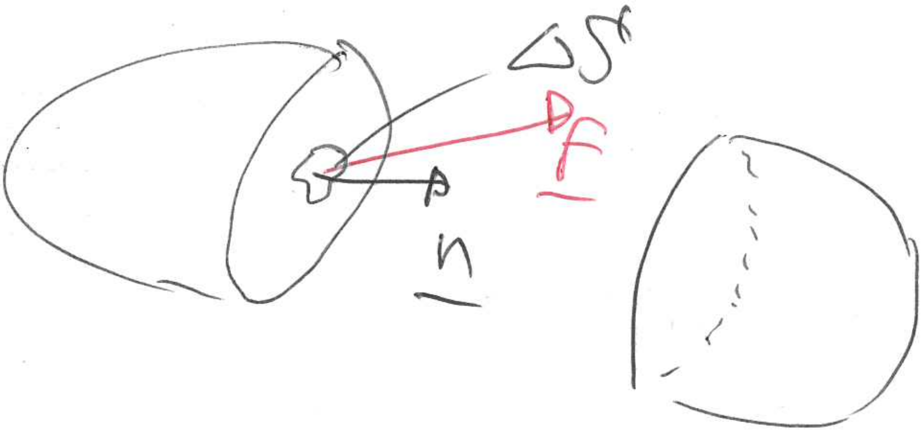
Every patch ΔS (with outer unit normal $\underline{n}$) is subject to a resultant force $\underline{F}$.

Def: Traction

$$\boxed{\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S}}$$ is a vector!

$\underline{t}$ is the force per unit area exerted onto the fluid.

Similarly: cut the blob along a plane with normal $\underline{n}$:



$\underline{F}$ represents the force exerted onto ΔS by the "other half" of the blob.

Def: Stress

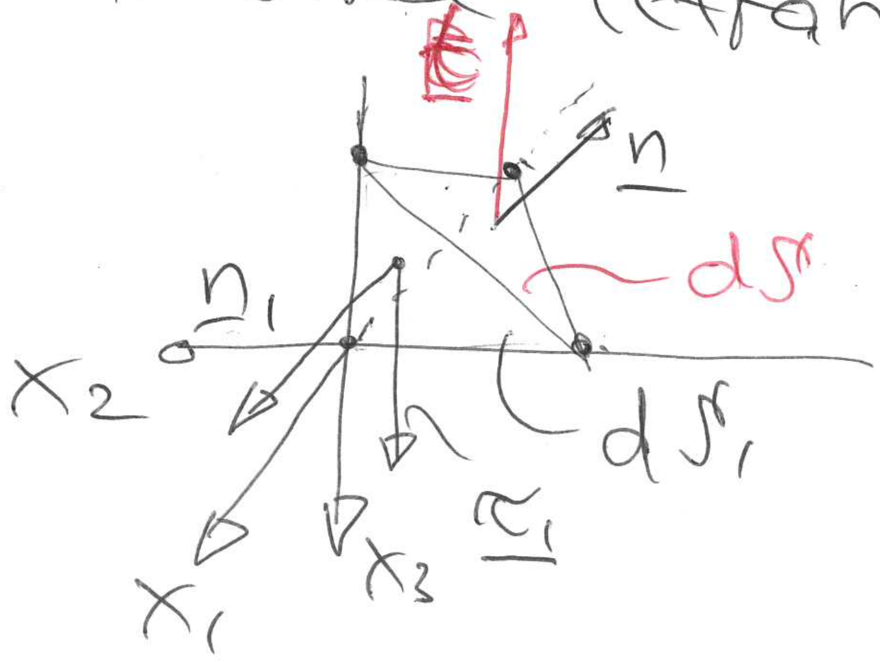
$$\boxed{\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S}}$$

Note: The stress depends on:

- the position in the fluid
- the direction $\underline{n}$ of the imaginary cut.

② The stress tensor

To examine the dependence on $\underline{n}$: consider an infinitesimal tetrahedron:



Face i is characterised by $x_i = \text{const}$ & $\underline{n_i} = \underline{e_i}$ & has area dS_i

Use area vectors to (6)
represent the faces
(i.e. their area & orientation)
as $dS_{(i)} \underline{n}_{(i)}$

Then:

$$\underline{n}_i \, dS_i + \underline{n} \, dS = \underline{0}$$

(Exercise)

$$\cdot \underline{n}_j = \underline{e}_j$$

$$\underline{n}_i \cdot \underline{n}_j \, dS_i + \underline{n} \cdot \underline{n}_j \, dS = 0$$

$$\underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}} \, dS_i + \underline{n} \cdot \underline{e}_j \, dS = 0$$

$$\underbrace{\delta_{ij} \, dS_i}_{dS_j} + n_j \, dS = 0$$

$$\underline{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

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$$dS_j = -n_j \, dS$$

<u>Now</u>: Balance of forces
(neglect body forces, accel....)

$$\underline{t}\, dS = -\underline{\tau}_j \, dS_j$$

$$\underline{t}\, \cancel{dS} = +\underline{\tau}_j \, n_j \, \cancel{dS}$$

$$\underline{t} = \underline{\tau}_j \, n_j$$

in components

$$\boxed{t_i = \tau_{ij} \, n_j}$$