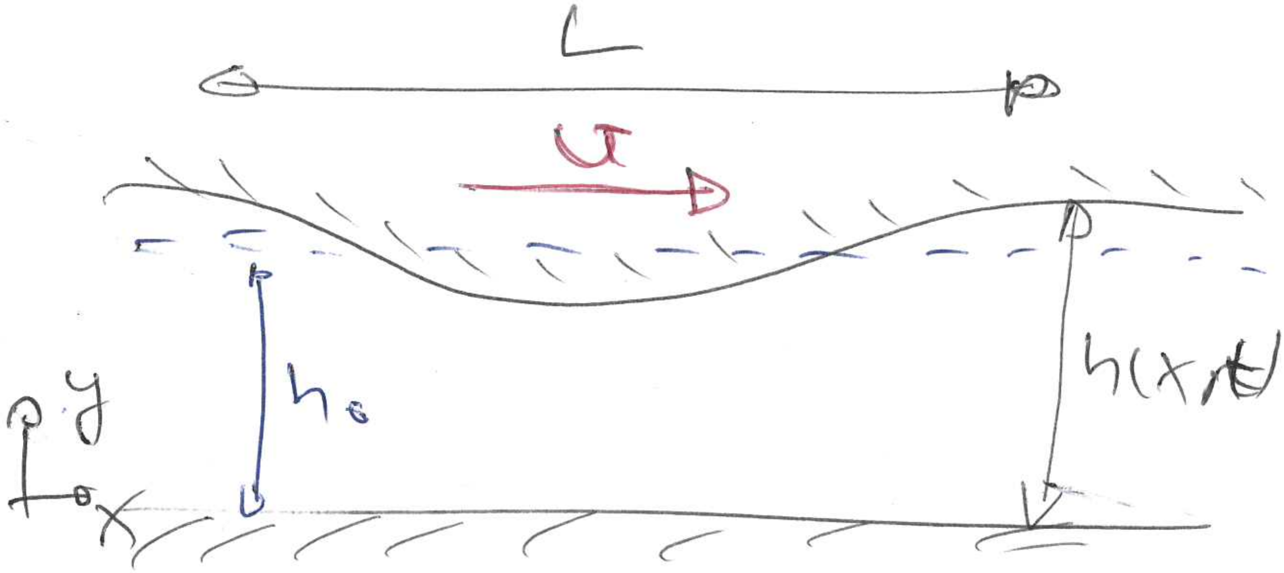



$x = L\tilde{x}$

$y = h_0\tilde{y}$

$u = U\tilde{u}$

$v = V\tilde{v} = \left(\frac{h_0}{L}\right)U\tilde{v}$

$t = \frac{L}{U}\tilde{t}$

$p = P\tilde{p}$

$h_0 \ll L \qquad \left|\frac{\partial h}{\partial x}\right| = O\left(\frac{h_0}{L}\right) \ll 1$

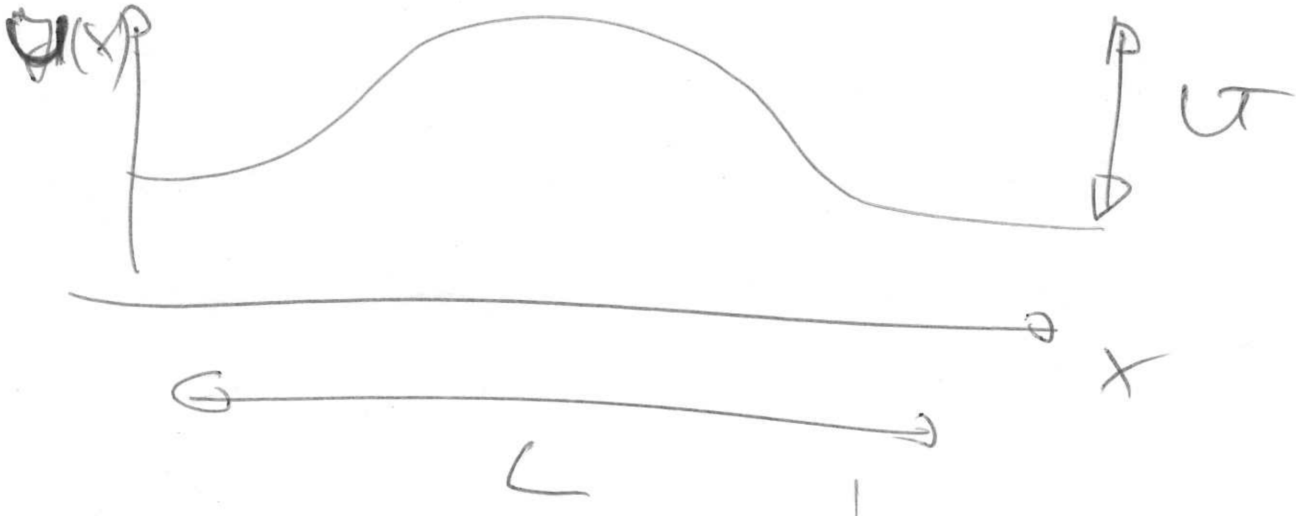
<u>x-NSE:</u>

$$\underbrace{\frac{\rho U h_0}{\mu}}_{Re}\underbrace{\frac{h_0}{L}}_{\ll 1}\frac{D\tilde{u}}{D\tilde{t}} = -\frac{P}{\left(\frac{\mu U}{h_0}\right)}\left(\frac{h_0}{L}\right)\frac{\partial\tilde{p}}{\partial\tilde{x}} + \left[\cancel{\left(\frac{h_0}{L}\right)^2\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}} + \frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right]$$

So if $Re\left(\frac{h_0}{L}\right) \ll 1$:

$$0 = -\frac{P}{\left(\frac{\mu U}{h_0}\right)}\left(\frac{h_0}{L}\right)\frac{\partial\tilde{p}}{\partial\tilde{x}} + \frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}$$

Aside: $u(x)$

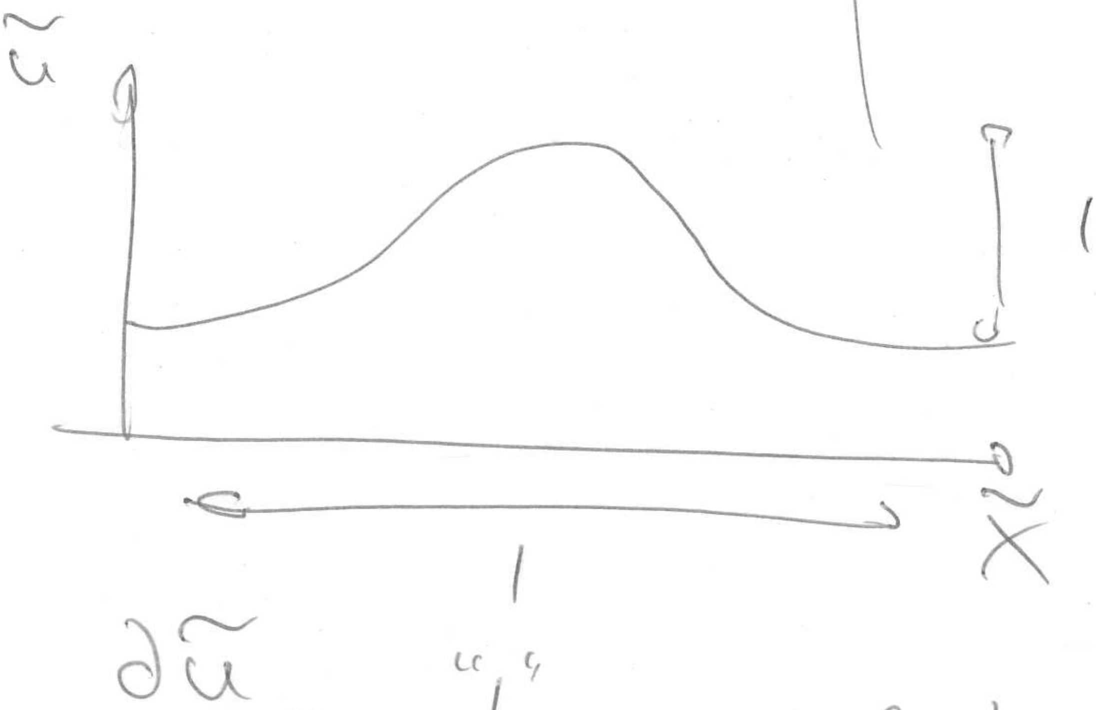



$$\frac{\partial u}{\partial x} = \,?$$

$$u = U\,\tilde{u}$$

$$x = L\,\tilde{x}$$

$$\frac{\partial u}{\partial x} = \frac{U}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}}$$

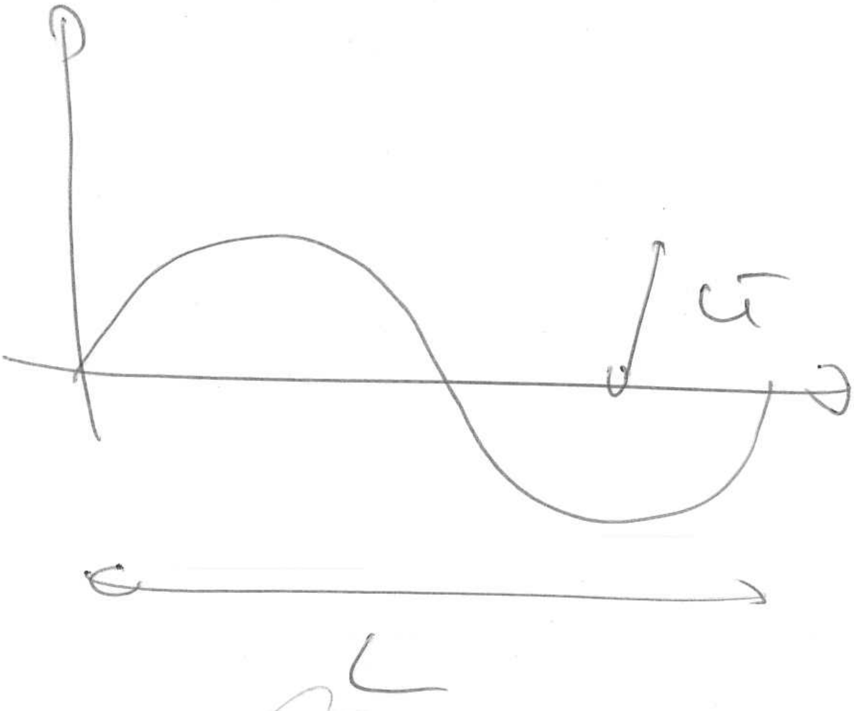



$$\frac{\partial \tilde{u}}{\partial \tilde{x}} = \frac{\text{“}1\text{”}}{\text{“}1\text{”}} = O(1)$$

(3

Specific example.

$$u(x) = U \sin\left(\frac{2\pi x}{L}\right)$$




$$\frac{du}{dx} = \frac{U}{L} 2\pi \underbrace{\cos\left(2\pi\left(\frac{x}{L}\right)\right)}_{O(1)}$$

$$\frac{du}{dx} = O\left(\frac{U}{L}\right)$$

$\boxed{\text{End aside}}$

This leaves a balance between the press. gradient & viscous resistence. The terms balance if

$$P = \underbrace{\frac{\mu U}{h_0}}_{\substack{\text{typical}\\ \text{shear}\\ \text{stress}}} \cdot \underbrace{\frac{1}{\left(\frac{h_0}{L}\right)}}_{\gg 1}$$

$$\boxed{0 = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}}$$

Exercise:

y-mom. eqn

$$\boxed{\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0}$$

= parallel flow eqns!

Back to dim. form.

$$\boxed{\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \quad &(1) \\ \frac{\partial p}{\partial y} &= 0 \quad &(2) \end{aligned}}$$

BC: ~~u(x)~~ $u(y=0) = 0$

$u(y = h(x,t)) = U$

Integrate (1) w.r.t. $y$:

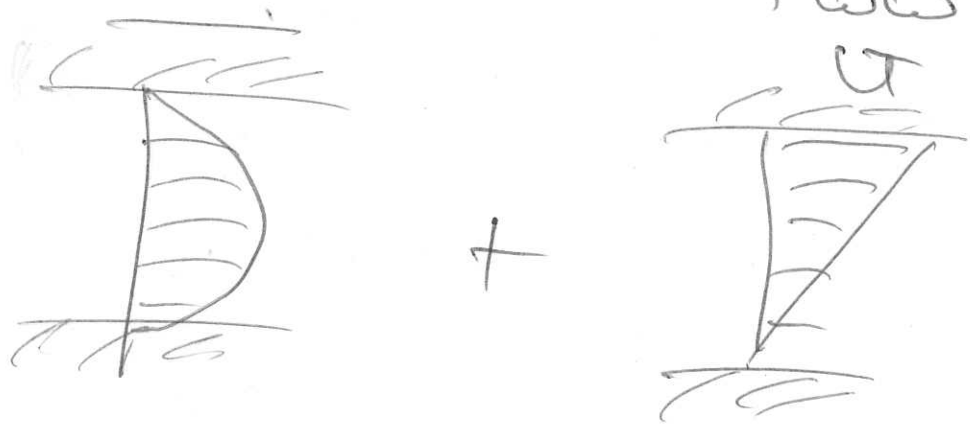
$$u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + Ay + B$$

const.

~~Rep~~ Apply BC.

(6)

$$\boxed{u(x,y,t) = \underbrace{\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - hy\right)}_{\text{press driven flow}} + \underbrace{U\frac{y}{h}}_{\text{shear flow}}} \quad (*)$$




$h(x,t)$

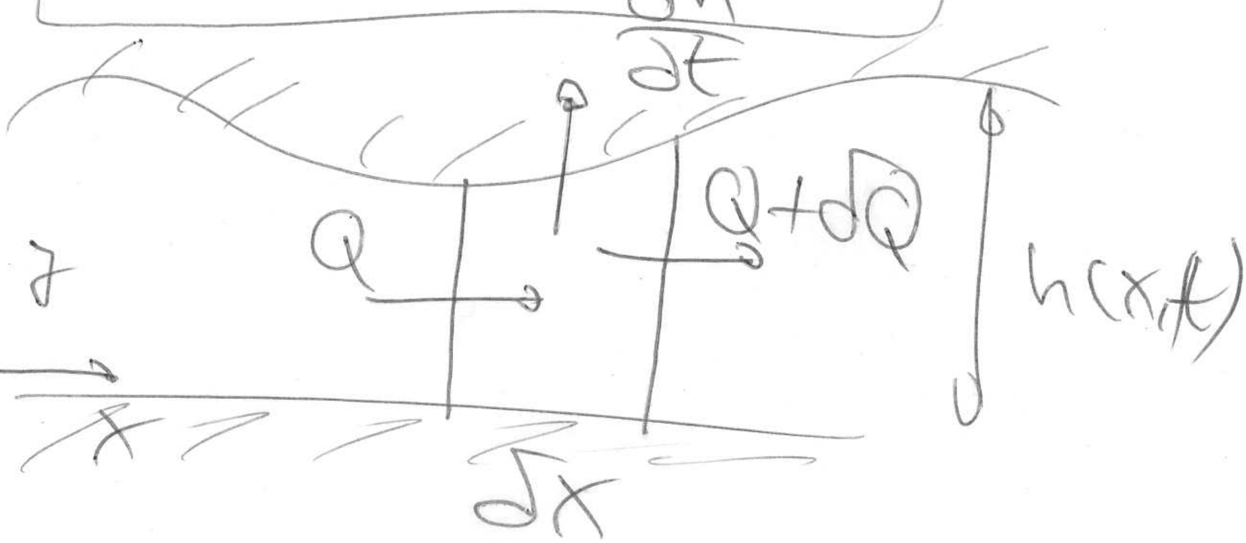
$\Rightarrow$ Flow is parallel flow in a channel with the local width.

<u>BUT</u>: what is $\frac{dp}{dx}$ ?

Mass conservation has not yet been enforced!

Consider volume flux through arbitrary cross section:

$$\boxed{Q(x) = \int_0^{h(x)} u \, dy}$$




Net outflow:

$$\delta Q + \frac{\partial h}{\partial t}\delta x = 0$$

as $\delta x \to 0$,

$$\boxed{\frac{\partial Q}{\partial x} = -\frac{\partial h}{\partial t}}$$

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$$Q(x) = \int_0^h u \, dy \longrightarrow (*)$$

⋮

$$Q(x) = -\frac{1}{12\mu}\frac{\partial p}{\partial x} h^3 + \frac{1}{2} U h$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{12\mu}\frac{\partial p}{\partial x} h^3 + \frac{1}{2} U h\right) = -\frac{\partial h}{\partial t}$$

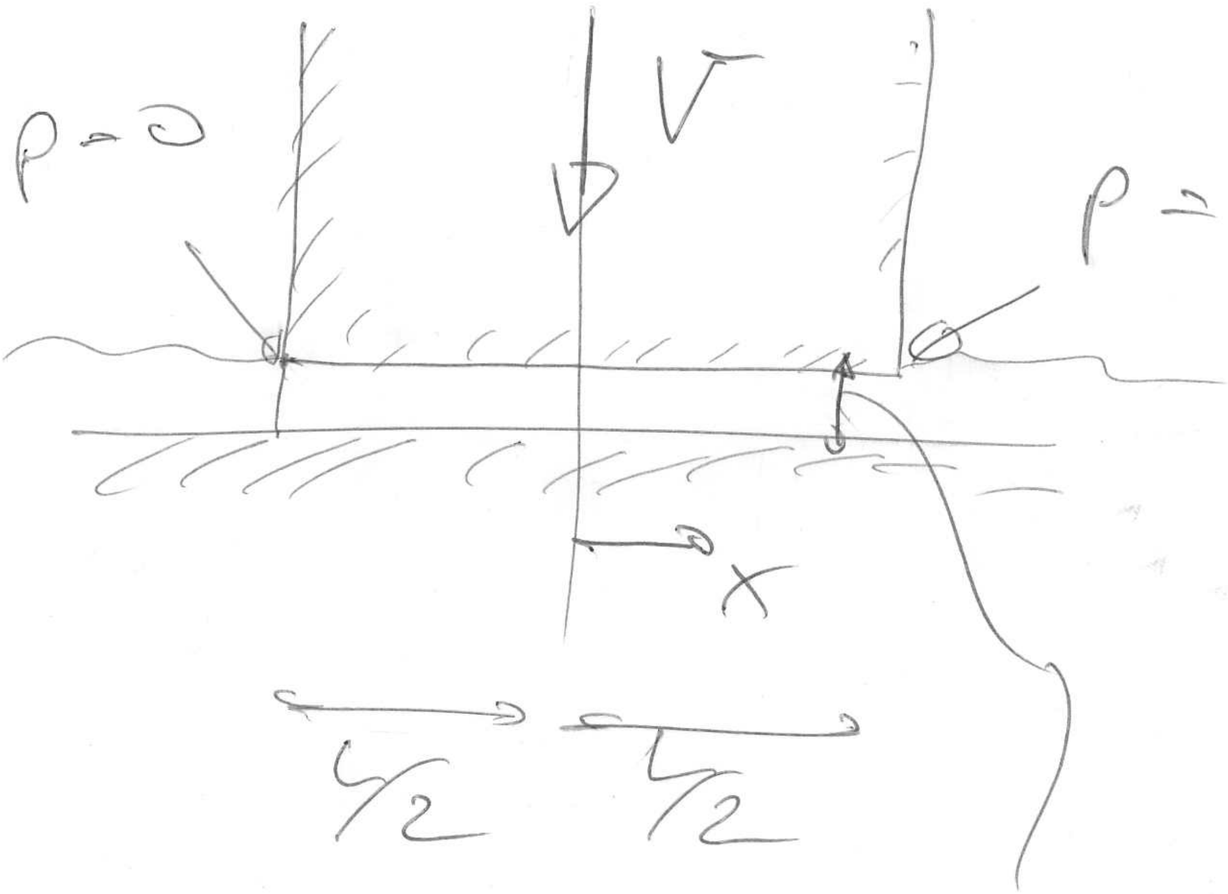
$$\boxed{\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right) = 12\frac{\partial h}{\partial t} + 6U\frac{\partial h}{\partial x}}$$

Reynold's eqn. of lubrication.

This is an eqn. for $p(x,t)$

Veloc. follows from (*).

Example: Squeeze film:




$h(x,t) = h_0 - Vt$

$h_0 \ll L$.

$\frac{\partial h}{\partial x} = 0$

$\frac{dh}{dt} = -V$

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{dp}{dx}\right) = 12\underbrace{\frac{dh}{dt}}_{-V}$$

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$$\frac{h^3}{\mu}\frac{\partial p}{\partial x} = -12Vx + \tilde{A}(t)$$

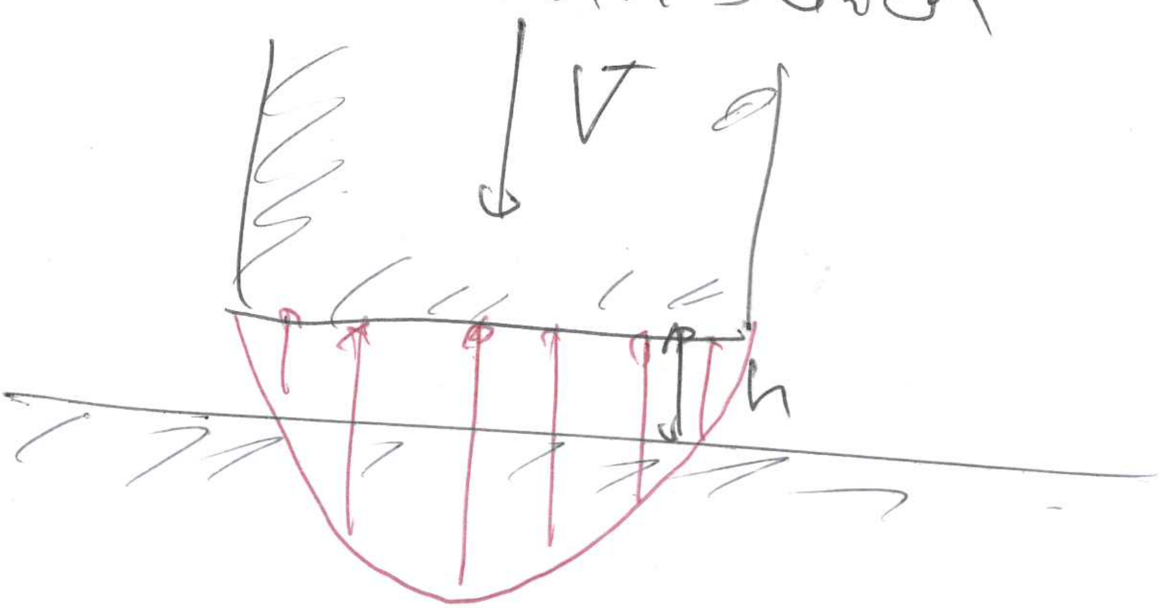
$$\frac{\partial p}{\partial x} = -\frac{12V\mu}{h^3}x + A(t)$$

$$p(x,t) = -\frac{12V\mu}{h^3}\frac{x^2}{2} + A(t)x + B(t)$$

<u>BC:</u> $p(x = \pm\frac{L}{2}, t) = 0$

$$\boxed{p(x,t) = -\frac{6V\mu}{h^3}\left(x^2 - \left(\frac{L}{2}\right)^2\right)}$$

Note: Parabolic press distribution



**Note:**

$p \to \infty$ as $h \to 0$.