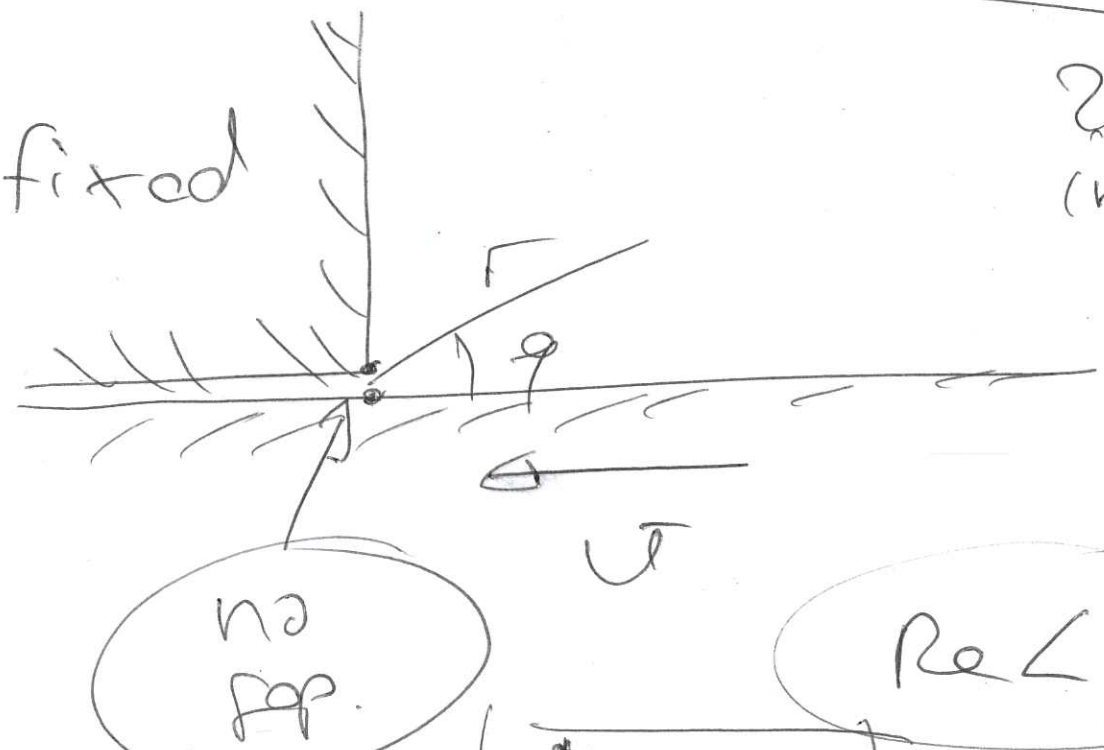
$$\cancel{\rho \frac{D\underline{u}}{Dt}} = -\nabla p + \mu \nabla^2 \underline{u}$$

1

$$\cancel{Re \frac{D\tilde{\underline{u}}}{D\tilde{t}}} = -\nabla \tilde{p} + \nabla^2 \tilde{\underline{u}}$$



2D incompr.

Re << 1

$$\boxed{\nabla^4 \psi = 0}$$

$\varphi = 0: \quad u = u_\varphi = 0 \Rightarrow \quad \psi(\varphi = 0) = 0 \quad (1)$

$\varphi = \frac{\pi}{2} \quad u = u_\varphi = 0 \Rightarrow \quad \psi(\varphi = \frac{\pi}{2}) = 0 \quad (2)$

$\varphi = 0: \quad u = u_r = -U = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad (3)$

$\varphi = \frac{\pi}{2} \quad u = u_r = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad (4)$

Want $\underbrace{\psi(r,\varphi) = g(r) f(\varphi)}_{\text{Ansatz!}}$

Now BC (3) & (4) show that

$\frac{1}{r}\frac{\partial \psi}{\partial \varphi}$ should be indep. of r

for fixed φ.

<u>Try</u>: $g(r) = u r$

$\psi(r,\varphi) = u r f(\varphi)$

BC

$$\varphi = 0: \ \psi = 0 \Rightarrow f(0) = 0$$

$$\varphi = \frac{\pi}{2} \ \psi = 0 \Rightarrow f(\tfrac{\pi}{2}) = 0$$

$$\varphi = 0: \ \frac{1}{r}\frac{\partial \psi}{\partial \varphi} = -u \Rightarrow f'(0) = -1$$

$$\varphi = \frac{\pi}{2}: \ \frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0 \Rightarrow f'(\tfrac{\pi}{2}) = 0$$

PDE: $\nabla^4\psi = \nabla^2 \underbrace{\nabla^2\psi}_{U r f(\varphi)} = 0$

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) U r f(\varphi)$$

$$= \frac{U}{r} f + \frac{U}{r} f''$$

$$\nabla^4\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) U r^{-1}(f'' + f)$$

$$= U\left\{(-1)(-2) r^{-3}(f + f'') - r^{-3}(f + f'') + r^{-3}(f'' + f^{IV})\right\}$$

$$\nabla^4\psi = \frac{U}{r^3}\left(f(\varphi)(2-1) + f''(\varphi)(2-1+1) + f^{IV}(\varphi)\right)$$

$$\nabla^4\psi = \frac{U}{r^3}\underbrace{\left(f + 2f'' + f^{IV}\right)}_{=0} = 0$$

$f(\varphi) \sim e^{\lambda\varphi}$

char. poly:

$$1 + 2\lambda^2 + \lambda^4 = 0$$

$$(\lambda^2 + 1)^2 = 0$$

$$\lambda_{1234} = \pm i$$

two repeated roots
each one gives $e^{\pm i\varphi}$
or $\sin\varphi / \cos\varphi$

Gen. soln:

$$f(\varphi) = A\sin\varphi + B\cos\varphi + C\varphi\sin\varphi + D\varphi\cos\varphi$$

for 4 constants A, B, C, D
from 4 BC.

⋮

(5)

$$\boxed{\psi(r,\varphi) = \frac{Ur}{\left(\frac{\pi}{2}\right)^2 - 1}\left(-\left(\frac{\pi}{2}\right)^2 \sin\varphi + \varphi\cos\varphi + \frac{\pi}{2}\varphi\sin\varphi\right)}$$

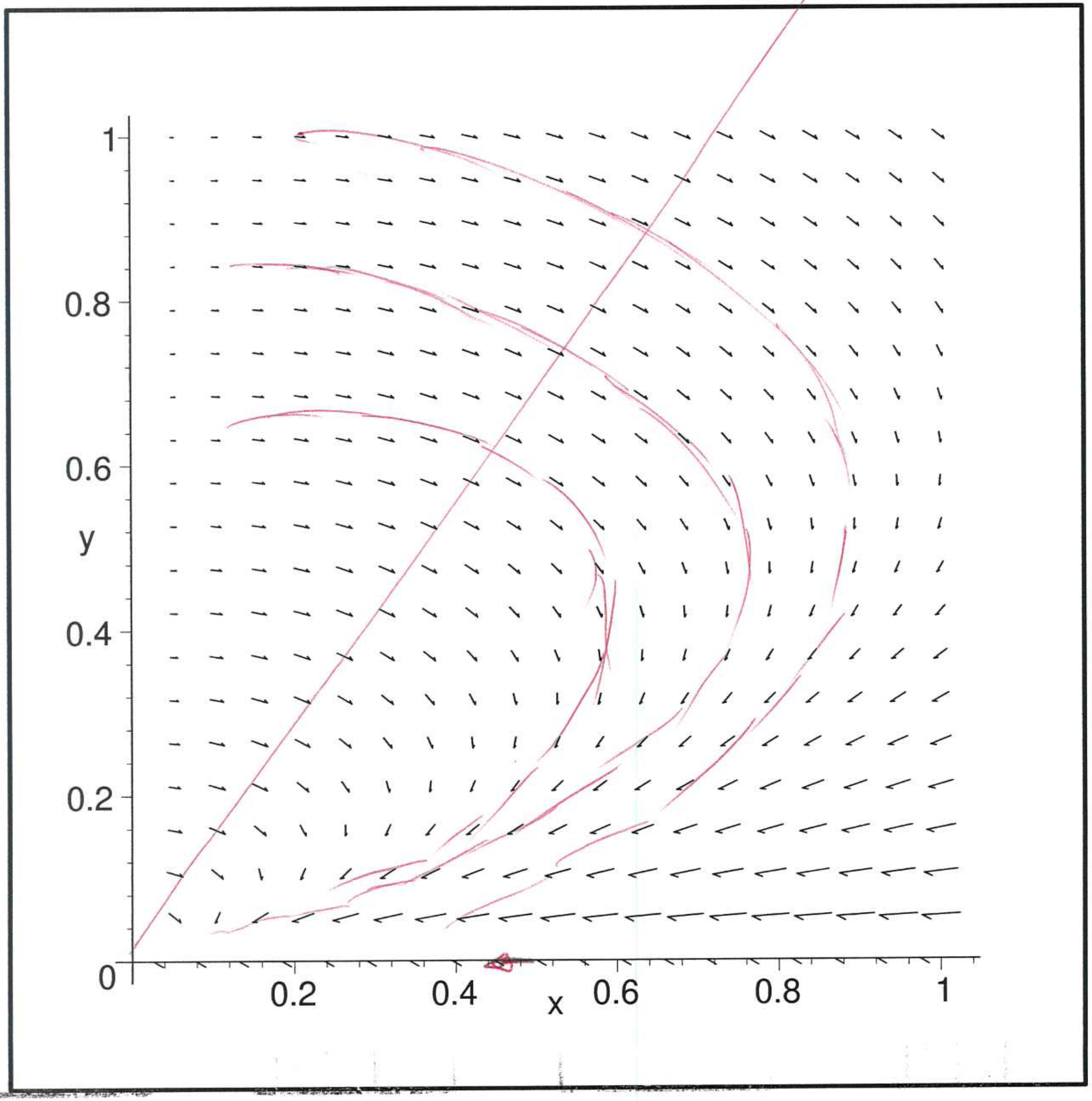
Veloc. are given by derivs:

$u_\varphi = v = -\frac{\partial\psi}{\partial r}$ is indep. of r.

$u_r = u = \frac{1}{r}\frac{\partial\psi}{\partial\varphi}$ is indep of r.

Dîsela

Velocity field for scraping flow at zero Reynolds number:
The vertical wall at x=0 is stationary. The horizontal wall at y=0 moves to the left with unit velocity.



6

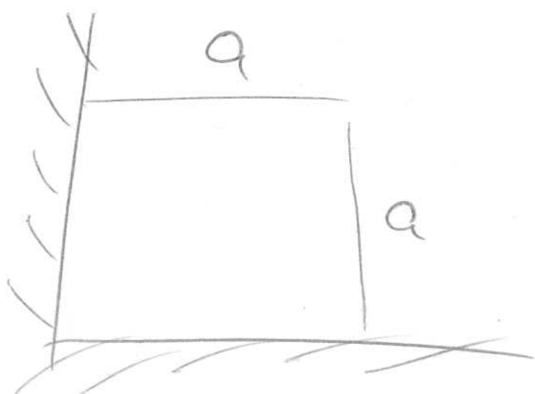
Discussion:

(I) Nonuniformity of the solution

we have assumed that the flow is slow & viscous so that

$$Re \ll 1$$

$$Re = \frac{U a}{\nu}$$



Flow has no lengthscale! choose a as the size of the "box" near origin.

Given U & ν I

can choose a so that

$$Re = \frac{Ua}{\nu} \ll 1$$

$\Rightarrow$ Stokes solution is valid near the origin.

BUT for large distances from origin the Reynolds number will become large

$\Rightarrow$ Stokes eqn are no longer valid & we must solve the full N.St. eqns.

The solution we have found is a local soln. of the N.St. eqns.