$$Re = \frac{Ua}{\nu}$$

$$\cancel{Re \frac{D\underline{\tilde{\omega}}}{D\tilde{t}}} = \cancel{Re} (\underline{\tilde{\omega}} \cdot \nabla) \underline{\tilde{u}} + \nabla^2 \underline{\tilde{\omega}}$$

as $Re \to 0$:

$$\underline{0} = \nabla^2 \underline{\omega}$$

Also from N. St:

$$\cancel{Re \frac{D\underline{\tilde{u}}}{D\tilde{t}}} = -\nabla \tilde{p} + \nabla^2 \underline{\tilde{u}}$$

as $Re \to 0$:

Stokes eqns:

$$\underline{0} = -\nabla \tilde{p} + \nabla^2 \underline{\tilde{u}}$$

or in dimensional terms

$$\underline{0} = -\nabla p + \mu \nabla^2 \underline{u}$$

Take curl:

$$\nabla \times \underline{0} = -\underbrace{\nabla \times \nabla \tilde{p}}_{} + \nabla^2 \underbrace{\nabla \times \tilde{\underline{u}}}_{\tilde{\underline{\omega}}}$$

$$\boxed{\underline{0} = \nabla^2 \tilde{\underline{\omega}}}$$ as before!

In 2D: $\underline{\omega} = \omega \underline{e}_z$

$$0 = \nabla^2 \omega$$

and: $\omega = -\nabla^2 \psi$

$$\boxed{\nabla^4 \psi = 0}$$ for $Re = 0$

where

$$\nabla^4 = \nabla^2 \nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

$$\boxed{\nabla^4 \psi = 0}$$

is the biharmonic PDE:
4^{th} order linear PDE for ψ.

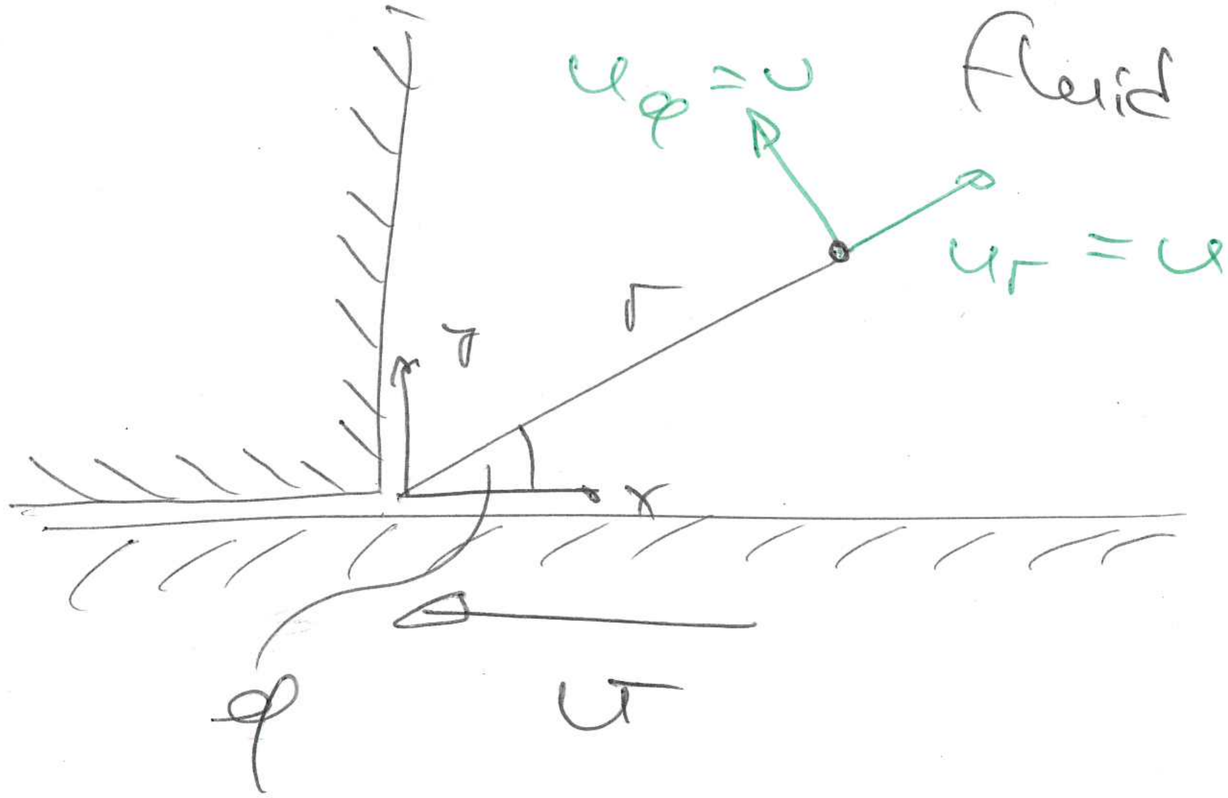
As post-processing:

$$\omega = -\nabla^2 \psi$$

$$u = \frac{\partial \psi}{\partial y} \; ; \quad v = -\frac{\partial \psi}{\partial x}$$

Stokes flow example

Scraping flow:



Assume: slow, steady viscous flow so that the Stokes eqns are valid:

$$\boxed{\nabla^4 \psi = 0}$$

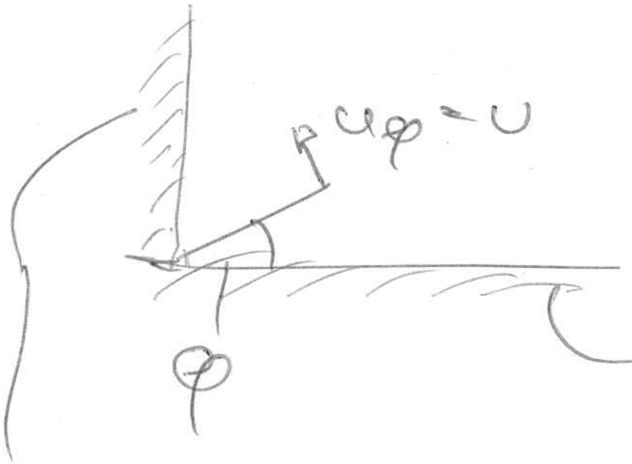
$$\nabla^4 = \nabla^2 \nabla^2$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

$$u_r = u = \frac{1}{r}\frac{\partial \psi}{\partial \varphi}$$

$$u_\varphi = v = -\frac{\partial \psi}{\partial r}$$

BC: Impermeability:



$\varphi = 0$: $v = 0$

$\varphi = \frac{\pi}{2}$: $v = 0$

$\varphi = 0$:

$$\underbrace{v = -\frac{\partial \psi}{\partial r} = 0 \quad \forall r}_{\psi = \text{const} = C_1 \text{ at } \varphi = 0}$$

$\varphi = \frac{\pi}{2}$:

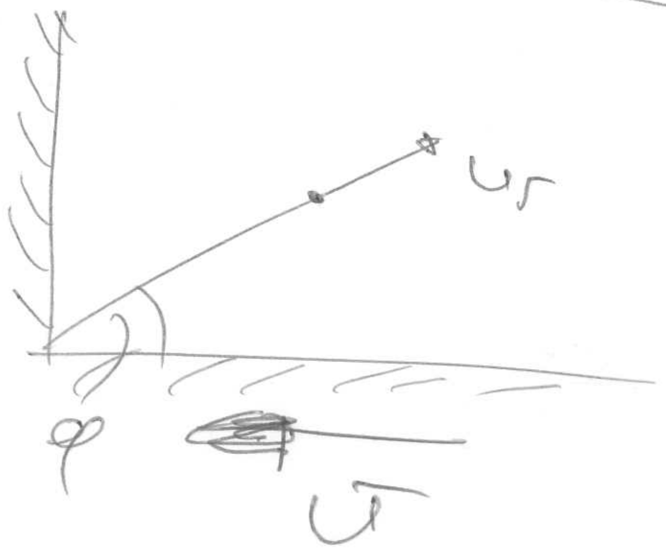
$$\underbrace{v = -\frac{\partial \psi}{\partial r} = 0}_{\psi = \text{const} = C_2 \text{ at } \varphi = \frac{\pi}{2}} \quad \forall r$$

we must have $C_1 = C_2$ because if not the jump in the streamfct "across the corner" would imply that fluid passes into that corner.

$$C_1 = C_2 = 0$$

$$\boxed{\begin{aligned} \psi(\varphi = 0) &= 0 \quad (1) \\ \psi\left(\varphi = \frac{\pi}{2}\right) &= 0 \quad (2) \end{aligned}}$$

<u>No slip:</u>



$\varphi = 0$: $u_r = u = -U$

$\varphi = \frac{\pi}{2}$: $u_r = u = 0$

$$\varphi = 0: \quad \boxed{\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = -U \quad \forall r} \quad (3)$$

$$\varphi = \frac{\pi}{2}: \quad \boxed{\frac{1}{r}\frac{\partial \psi}{\partial \varphi} = 0 \quad \forall r} \quad (4)$$