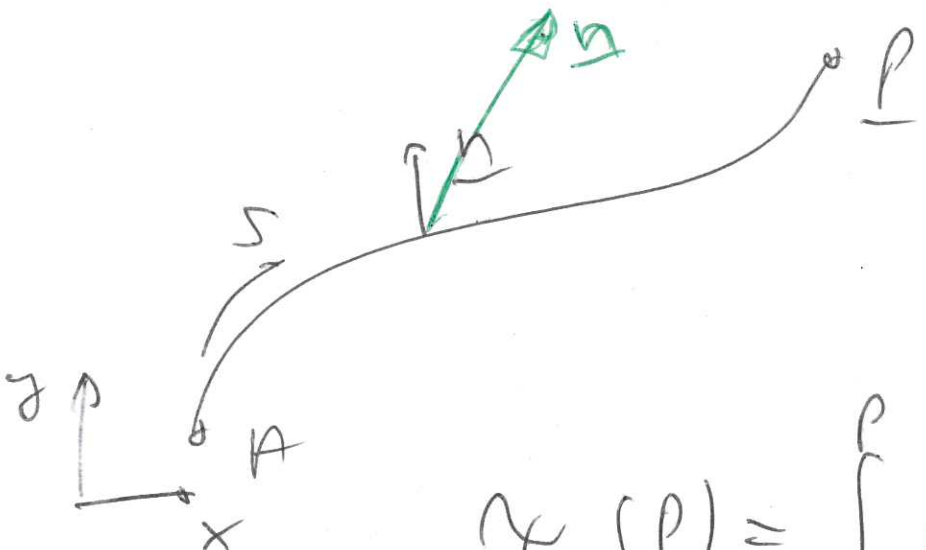


$\boxed{\text{2D incompr.}}$

$$\psi_A(P) = \int_A^P \underline{u} \cdot \underline{n} \, ds$$

$$\Downarrow$$

$$\underbrace{\boxed{u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}}}^{\psi(x,y)}$$

Similar to potentials, or Airy stress fcts, etc.

Remarks:

(1) Derivation involved the integral form of the continuity eqn
$\Longrightarrow$ $\nabla \cdot \underline{u} = 0$ should be satisfied automatically.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \overset{?}{=} 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \overset{?}{=} 0$$

If we formulate the problem in terms of a stream fct we can ignore the continuity eqn.

(2) Impermeable boundaries (again)

e.g.:

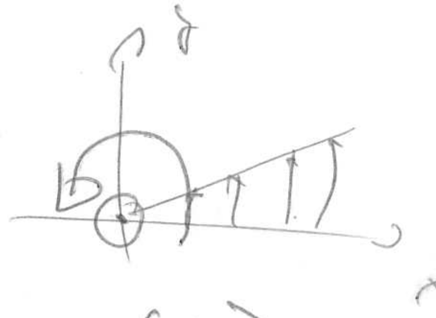
$$v = 0 = \frac{\partial \psi}{\partial x} \quad \forall x$$

$$\Rightarrow \psi = \text{const along } y = 0$$

(3) Relation to vorticity 2D

$$\underline{\omega} = \omega_z \underline{e_z} = \omega \underline{e_z}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$



$\omega > 0$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\boxed{\omega = -\nabla^2 \psi}$$

ψ "gets rid of continuity eqn"; can we formulate the N.St. eqns in terms of ψ & ω?

The Vorticity equation

- Use of the streamfunction allows us to automatically satisfy the continuity equation. Now we will try to transform the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

into an equation for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

- Here are a few results that we will use in the derivation:

$$\frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}), \tag{1}$$

$$\nabla \times \nabla \phi = 0, \tag{2}$$

and

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} + \mathbf{u}\underbrace{\nabla \cdot \boldsymbol{\omega}}_{0} - \boldsymbol{\omega}\underbrace{\nabla \cdot \mathbf{u}}_{0}, \tag{3}$$

where the last two terms vanish because

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u}) = \operatorname{div}\operatorname{curl}\mathbf{u} = 0$$

and

$$\nabla \cdot \mathbf{u} = 0.$$

(5)

- First, we use (1) in the momentum equation to obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$$

i.e.

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\boldsymbol{\omega}}_{LHS} = \underbrace{-\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}}_{RHS}$$

- Now take the curl of the LHS:

$$\nabla\times LHS = \frac{\partial}{\partial t}\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}} + \frac{1}{2}\underbrace{\nabla\times\nabla(\mathbf{u}\cdot\mathbf{u})}_{0\text{ because of (2)}} - \underbrace{\nabla\times(\mathbf{u}\times\boldsymbol{\omega})}_{\text{see (3)}}$$

i.e.

$$\nabla\times LHS = \frac{\partial\boldsymbol{\omega}}{\partial t} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$$

- ...and the RHS:

$$\nabla\times RHS = -\frac{1}{\rho}\underbrace{\nabla\times\nabla p}_{0\text{ because of (2)}} + \nu\nabla^2\underbrace{(\nabla\times\mathbf{u})}_{\boldsymbol{\omega}}$$

- Now combine the remaining non-zero terms

$$\underbrace{\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}}_{D\boldsymbol{\omega}/Dt} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = \nu\nabla^2\boldsymbol{\omega}.$$

(6

- The resulting equation is the *vorticity transport equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \tag{4}$$

  which shows that the rate of change of the vorticity of material particles, $D\boldsymbol{\omega}/Dt$, is controlled by 'vortex stretching' (described by $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$; this is a familiar result from inviscid fluid mechanics) and by diffusion (described by $\nu\nabla^2\boldsymbol{\omega}$). The diffusion of vorticity only occurs in viscous flows.

- For 3D flows, the first term on the RHS in (4) represents vortex stretching: velocity gradients lead to a change in the rate of rotation of material particles.

- Note that for 2D flows, vortex stretching is absent since $\mathbf{u} = u(x, y)\ \mathbf{e}_x + v(x, y)\ \mathbf{e}_y$ and $\boldsymbol{\omega} = \omega(x, y)\ \mathbf{e}_z$ and therefore $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0$.

- The vorticity transport equation provides an interesting interpretation of the kinematic viscosity ν: The kinematic viscosity is the diffusion coefficient for the diffusion of vorticity.

- Many phenomena in viscous fluid mechanics can be interpreted in terms of the diffusion of vorticity but this is (unfortunately) beyond the scope of this course.

- For 2D flows, the vorticity transport equation

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{D\omega}{Dt} = \nu \nabla^2 \omega$$

  together with the equation for the vorticity in terms of the streamfunction

$$\omega = -\nabla^2 \psi$$

  and

$$u = \partial\psi/\partial y \quad \text{and} \quad v = -\partial\psi/\partial x$$

  provide the streamfunction-vorticity formulation of the Navier-Stokes equations, which consists of only two PDEs for the scalars ω and ψ rather than the three equations for u, v and p in the 'primitive variable' form.

- Scaling arguments show that in the limit of zero Reynolds number, only one fourth-order PDE for the streamfunction ψ needs to be solved, namely the biharmonic equation

$$\nabla^4 \psi = 0,$$

  where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

- This can also be shown directly by taking the curl of the Stokes equations.

vorticity transport eqn:

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2}$$

Assume scalings:

$$u_i = U \tilde{u}_i$$

$$x_j = a \tilde{x}_j$$

$$t = \frac{a}{U} \tilde{t}$$

$$\omega_j = \frac{U}{a} \tilde{\omega}_j$$

$$\left(\frac{U}{a}\right)^2 \frac{\partial \tilde{\omega}_i}{\partial \tilde{t}} + \frac{U^2}{a^2} \tilde{u}_j \frac{\partial \tilde{\omega}_i}{\partial \tilde{x}_j} =$$

$$\frac{U^2}{a^2} \tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{U}{a^3} \nu \frac{\partial^2 \tilde{\omega}_i}{\partial \tilde{x}_j^2} \qquad \Big| \cdot \frac{a^3}{U \nu}$$

$$\underbrace{\frac{Ua}{\nu}}_{Re} \underbrace{\left(\frac{D\tilde{\omega}_i}{D\tilde{t}}\right)}_{\frac{\partial \tilde{\omega}_i}{\partial \tilde{t}} + \tilde{u}_j \frac{\partial \tilde{\omega}_i}{\partial \tilde{x}_j}} = \underbrace{\frac{Ua}{\nu}}_{Re} \tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial^2 \tilde{\omega}_i}{\partial \tilde{x}_j^2}$$

where $Re = \frac{\rho U a}{\mu} = \frac{U a}{\nu}$

If $Re \ll 1$ (slow viscous flow on small lengthscales):

$$0 = \frac{\partial^2 \tilde{\omega}_i}{\partial \tilde{x}_j^2} = \tilde{\nabla}^2 \tilde{\omega}_i$$

$$0 = \nabla^2 \omega_i$$

$$0 = \nabla^2 \underline{\omega} \quad \text{for } Re \to 0$$