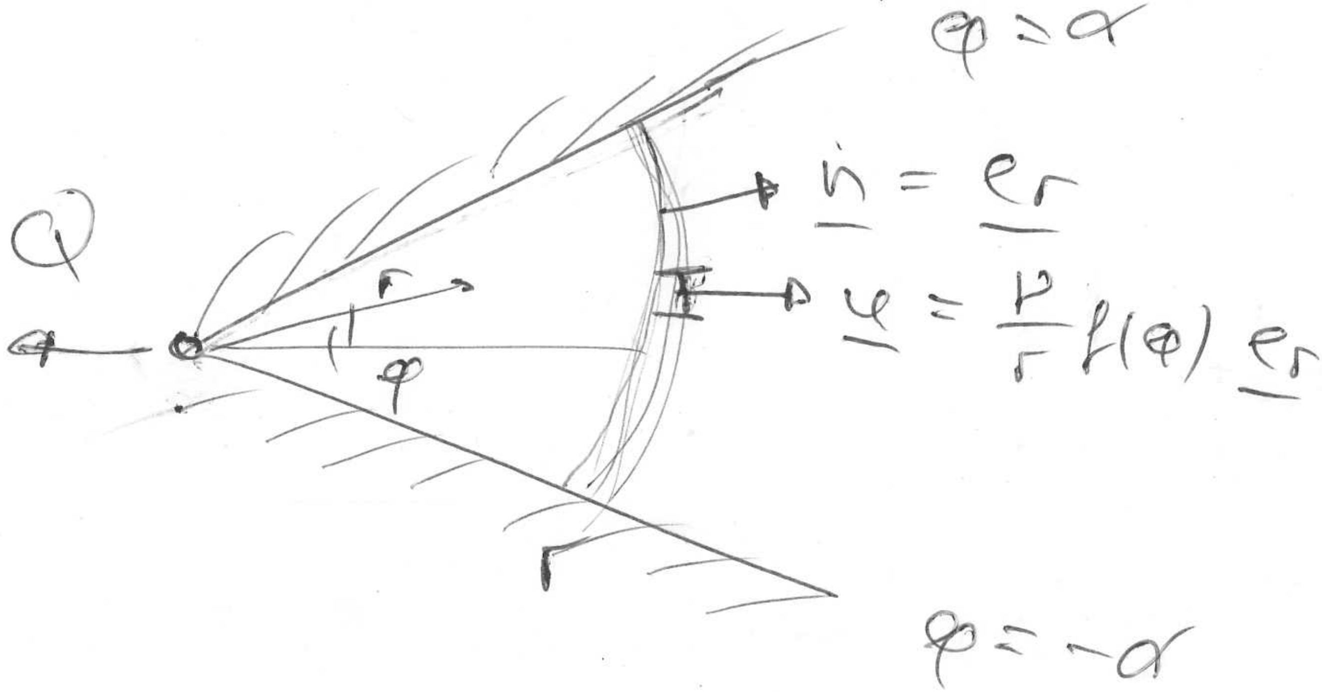


$$\boxed{\underline{u} = \frac{\nu}{r} f\left(\varphi; \alpha, \frac{Q}{\nu}\right) \underline{e_r}}$$

$$\boxed{f''' + 4f' + 2ff' = 0}$$

3^{rd} order nonlin ODE for $f(\varphi)$

$\underline{u}(\varphi = \pm\alpha) = \underline{0} \Rightarrow f(\varphi = \pm\alpha) = 0$

$$Q = -\int^{\alpha} \underline{u} \cdot \underbrace{\underline{n}}_{\underline{e_r}}\, r\, d\varphi$$

$$\boxed{Q = -\int_{\varphi=-\alpha}^{\alpha} \nu f(\varphi)\, d\varphi}$$

Streamfunction & vorticity eqns

Alternative formulation of N-St. especially suited for 2D flows.

$\boxed{\text{2D, incompressible}}$

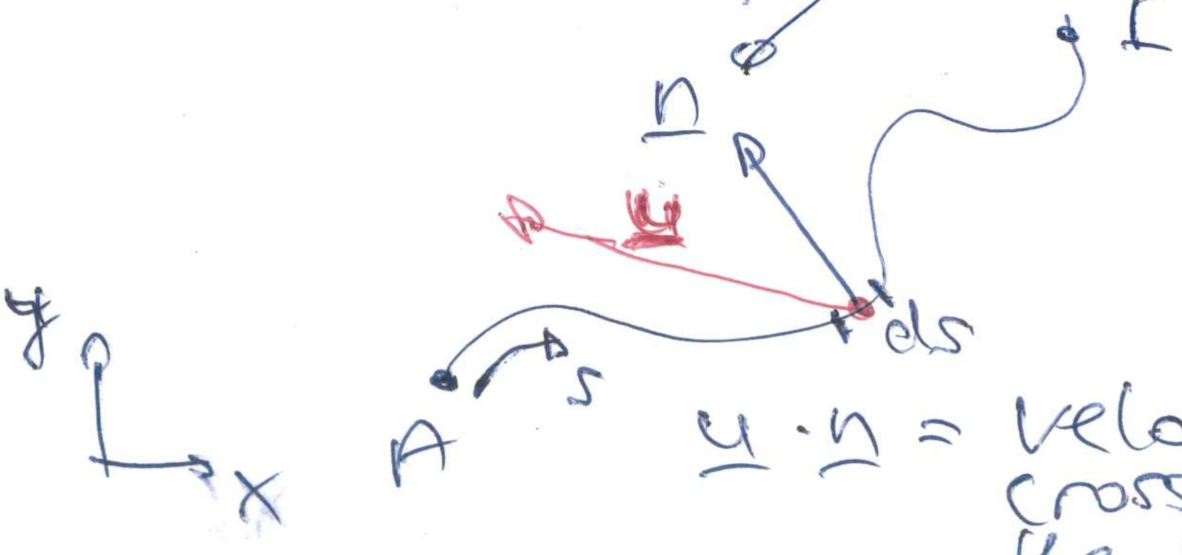
Stream fct:

$\underline{u} = u\,\underline{e_x} + v\,\underline{e_y}$

Def:

$$\psi_A(P) = \int_A^P \underline{u}\cdot\underline{n}\,ds$$

$\underline{n}$ to the left of path



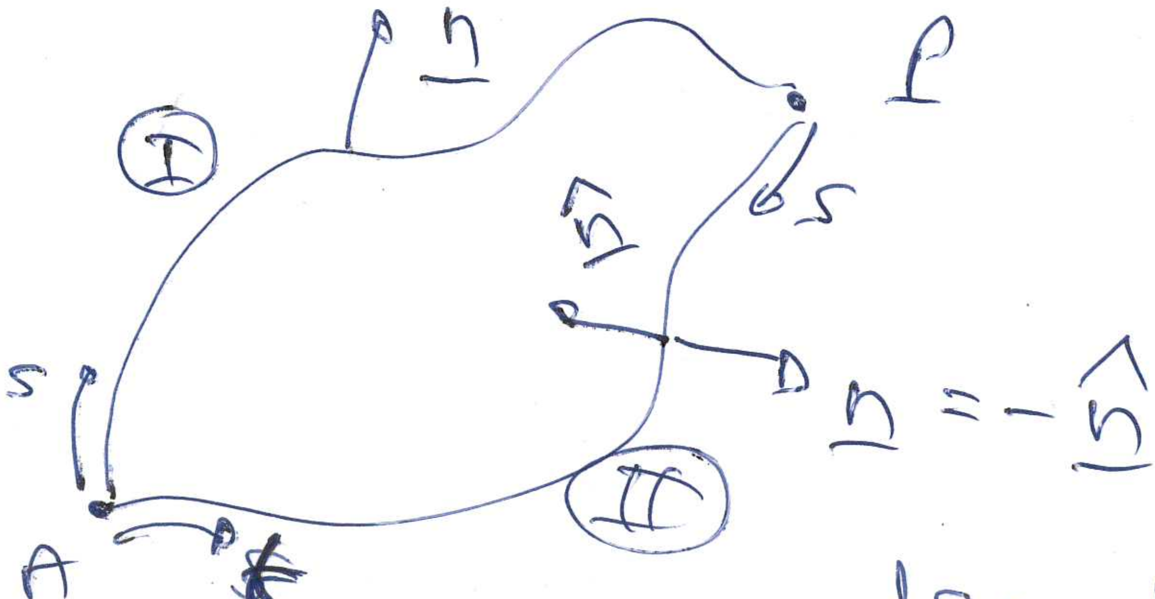
$\underline{u}\cdot\underline{n}$ = veloc. comp. crossing the line.

$\psi_A(P)$ represents volume flux (per unit depth in z-direction) crossing the line AP.

Implications

(1) $\psi_A(P)$ is path indep.

Proof:



$ds = -dt$

$$\psi_A^{\mathrm{I}}(P) = \int_{\mathrm{I}\,A}^{P} \underline{u}\cdot\underline{n}\, ds$$

$$\psi_A^{\mathrm{II}}(P) = \int_{\mathrm{II}\,A}^{P} \underline{u}\cdot\underbrace{\underline{\hat{n}}}_{(-\underline{n})}\, \underbrace{dt}_{(-ds)}$$

$$\psi_A^{(II)}(P) = -\int_P^A \underline{u}\cdot\underline{n}\,ds$$

(4

$$\psi_A^{(I)}(P) - \psi_A^{(II)}(P) = \int_{A\,(I)}^{P} \underline{u}\cdot\underline{n}\,ds + \int_{P\,(II)}^{A} \underline{u}\cdot\underline{n}\,ds$$

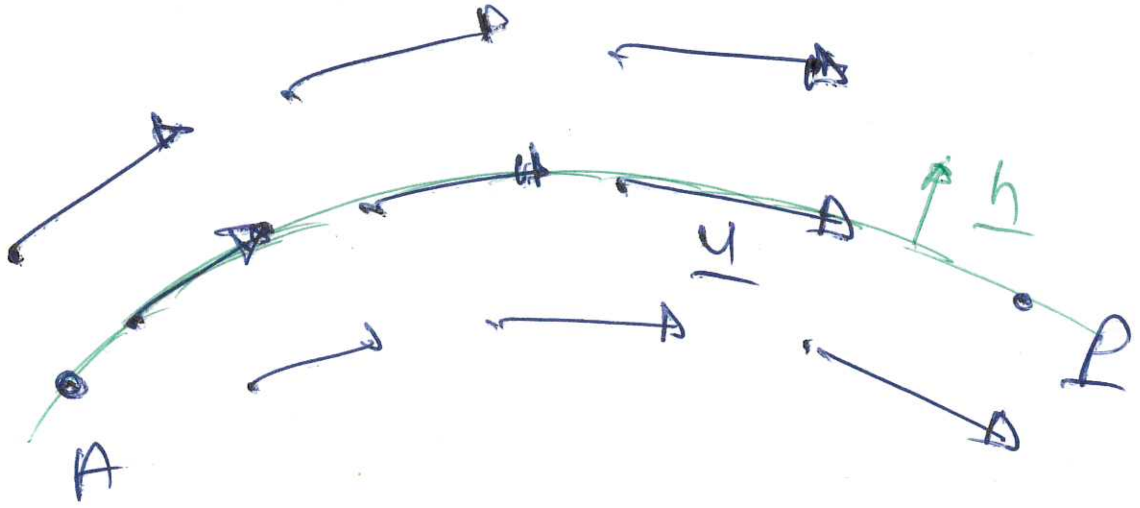
$$= \oint_{APA} \underline{u}\cdot\underline{n}\,ds = 0$$

(integral continuity eqn.)

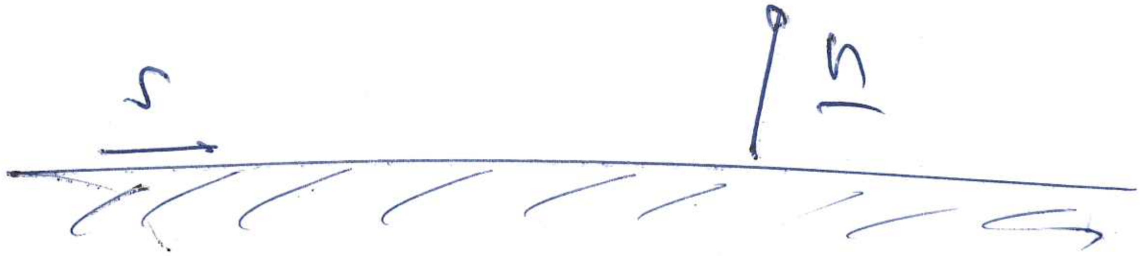
q.e.d.

(2) ψ is constant along streamlines.

Obvious, because streamlines are lines along which $\underline{u}\cdot\underline{n} = 0$ i.e. flow is tangential to these lines.



(3) impermeable boundaries are streamlines because $\underline{u} \cdot \underline{n} = 0$

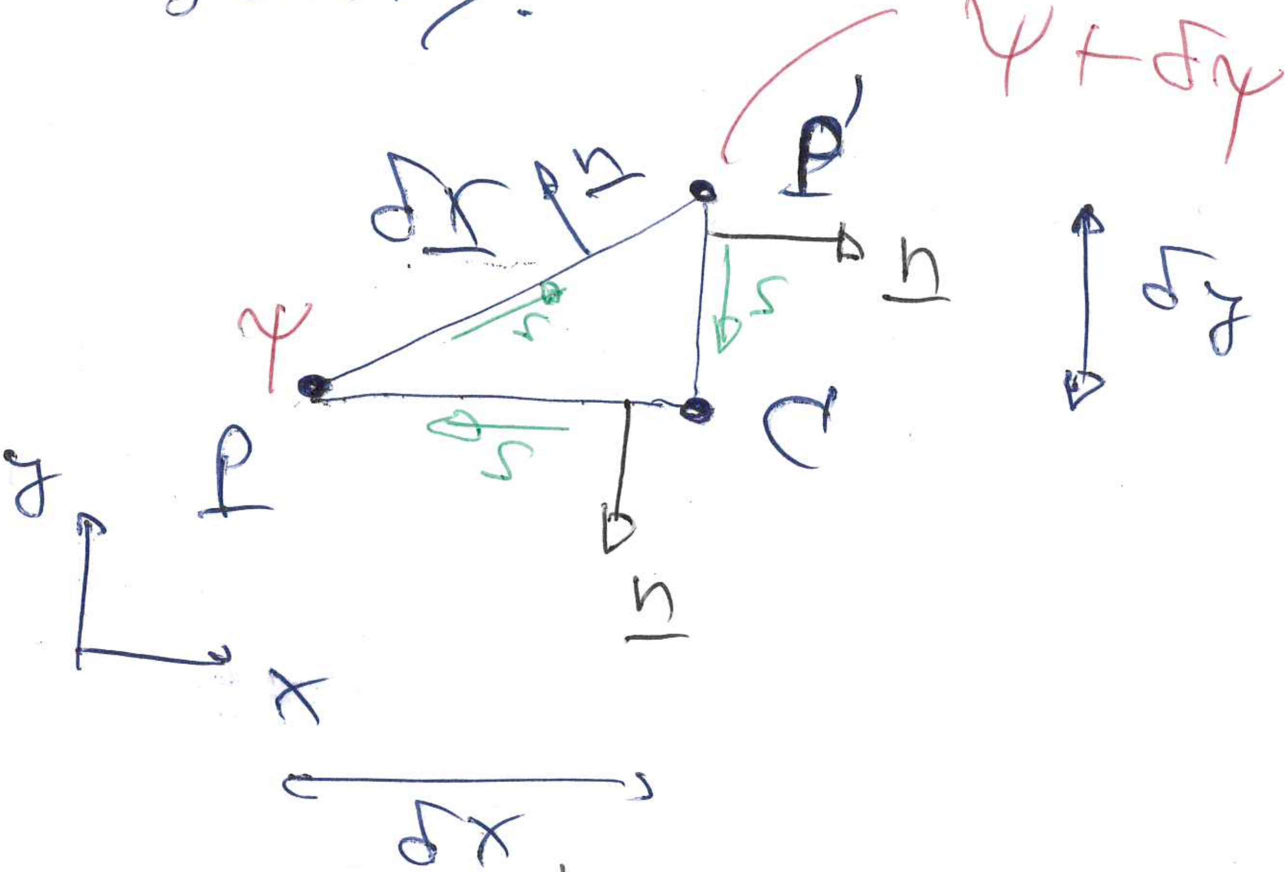


$\underline{u} = \underline{0}$

Common convention:

Set streamfct to zero on solid boundaries

What PDE does ψ satisfy?



$$\delta\psi = \int_P^{P'} \underline{u}\cdot\underline{n}\, ds$$

Continuity:

$$\oint_{\Delta} \underline{u}\cdot\underline{n}\, ds = 0$$

$$\delta\psi = \int_P^{P'} \underline{u}\cdot\underline{n}\, ds = -\int_{P'}^{C} \underbrace{\underline{u}\cdot\underbrace{\underline{n}}_{\underline{e}_x}}_{u}\,\underbrace{ds}_{-dy} - \int_{C}^{P} \underbrace{\underline{u}\cdot\underbrace{\underline{n}}_{-\underline{e}_y}}_{-v}\,\underbrace{ds}_{-dx}$$

$$\delta\psi = +\int_{P'}^{C} u\,dy - \int_{C}^{P} v\,dx$$

$\delta \underline{x} \to \underline{0}$ + mean value theorem

$$\delta\psi = u\,\delta y - v\,\delta x$$

Alternative $\psi(x,y)$

$$\delta\psi = \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y$$

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial\psi}{\partial x} &= -v \\ \frac{\partial\psi}{\partial y} &= u \end{aligned}}$$