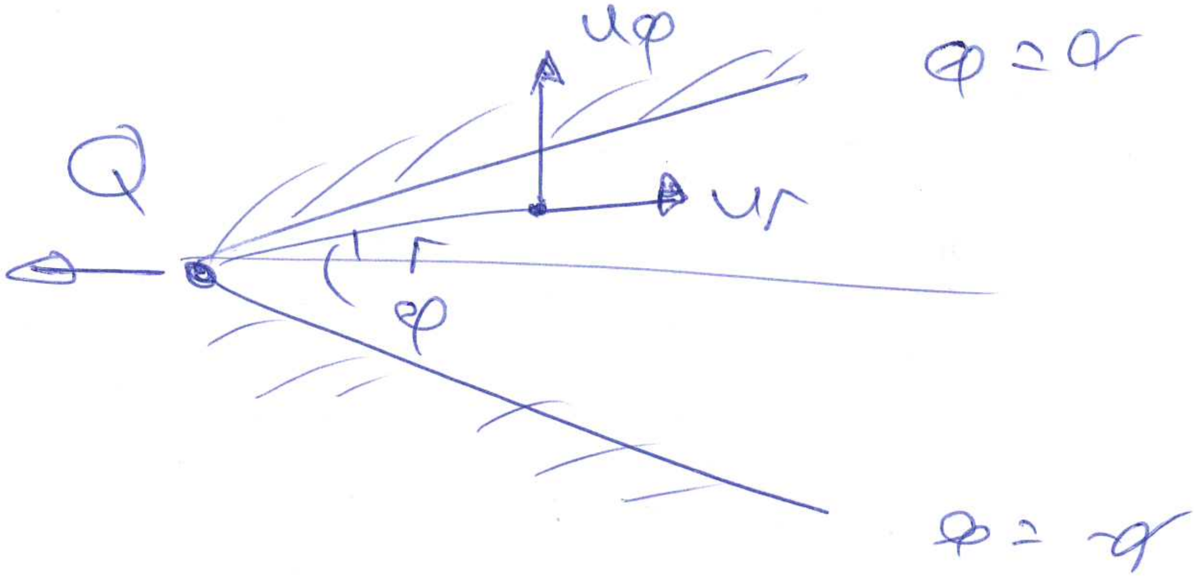


$$\underline{u} = \underline{u}(r, \varphi; \rho, \nu, Q, \alpha)$$

ρ	$\frac{kg}{m^3}$
ν	$\frac{m^2}{sec}$
Q	$= \frac{m^2}{sec}$
α	1
φ	1
r	m
$\underline{u}$	m/sec

To get the dimensions correct:

(2

$$\underline{u} = \underbrace{\frac{\nu}{r}} \underline{f}(r, \varphi; \rho, \nu, Q, \alpha)$$

this (or $\frac{Q}{r}$) produces the right dimensions.

$\Rightarrow$ $\underline{f}$ must be a dimensionless fct. of its arguments

- ρ is only quantity with kg $\Rightarrow$ soln cannot ~~depend~~ depend on it!

Also: $\underline{f}$ cannot depend on r because there is no other quantity which only ~~depends~~ has dimension "m".

$$\Rightarrow \quad \underline{u} = \frac{\nu}{r} \underline{f}\left(\varphi; \alpha, \frac{Q}{\nu}\right)$$

$$= \underbrace{\frac{\nu}{r} f_r\left(\varphi; \alpha, \frac{Q}{\nu}\right)}_{u_r} \underline{e_r} + \overbrace{\frac{\nu}{r} f_\varphi\left(\varphi; \alpha, \frac{Q}{\nu}\right)}^{u_\varphi} \underline{e_\varphi}$$

$u = u_r \qquad v = u_\varphi = 0$ (3

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \cancel{\frac{v^2}{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left[\cancel{\nabla^2 v} - \cancel{\frac{v}{r^2}} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right],$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0.$$

$$\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r f_r(\varphi)\right)} + \underbrace{\frac{1}{r}\frac{\partial}{\partial \varphi}\left(\frac{\nu}{r} f_\varphi(\varphi)\right)}=0$$

where

$$\Rightarrow \frac{\partial f_\varphi}{\partial \varphi} = 0$$

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

$f_\varphi = \text{const.}$

$\Rightarrow v = u_\varphi = \text{const.}$

$u_\varphi = \text{const.}$

Apply BC: $u_\varphi(\varphi = \alpha) = u_\varphi(\varphi = -\alpha) = 0$

$\Rightarrow u_\varphi = 0$

$\Rightarrow \underline{u}(r, \varphi; \dots) = u_r \underline{e_r} = u \, \underline{e_r}$

<u>r-mom. eqn:</u>

$$u \frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \overbrace{\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \frac{u}{r^2} \right)}^{\nabla^2 u}$$

$$u = \frac{\nu}{r} f(\varphi) = \nu r^{-1} f(\varphi)$$

$$\frac{\partial u}{\partial r} = -\nu r^{-2} f(\varphi)$$

$$\frac{\partial^2 u}{\partial r^2} = 2\nu r^{-3} f(\varphi)$$

$$\frac{\partial^2 u}{\partial \varphi^2} = \nu r^{-1} f''(\varphi)$$

$$\left(\frac{\nu}{r}f\right)\left(\frac{-\nu}{r^2}f\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\cancel{-\nu\frac{1}{r^3}f} + \cancel{\frac{2\nu}{r^3}f} + \frac{\nu}{r^3}f'' \cancel{-\frac{\nu}{r^3}f}\right)$$

$$\boxed{-\frac{\nu^2}{r^3}f^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu^2}{r^3}f''} \qquad (1)$$

<u>φ-mom. eqn:</u>

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + 2\nu\frac{1}{r^2}\frac{\partial u}{\partial \varphi}$$

$$\frac{\partial u}{\partial \varphi} = \frac{\nu}{r}f'(\varphi)$$

$$\frac{\partial p}{\partial \varphi} = \rho\nu\frac{2}{r}\frac{\nu}{r}f'(\varphi)$$

$$\boxed{p = \rho\frac{2\nu^2}{r^2}f(\varphi) + g(r)} \qquad (2)$$

into (1)

$$-\frac{\nu^2}{r^3}f^2 = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(\rho\frac{2\nu^2}{r^2}f(\theta)+g(r)\right)+\frac{\nu^2}{r^3}f''$$

$$-\frac{\nu^2}{r^3}f^2 = -\frac{1}{\rho}\left(\frac{dg}{dr}-4\rho\nu^2 r^{-3}f(\theta)\right)+\frac{\nu^2}{r^3}f''$$

All terms go like r^{-3}

$\Rightarrow \frac{dg}{dr} = Ar^{-3}$ for some constant A.

$$g(r) = -\frac{1}{2}Ar^{-2} + C$$

$$= \frac{B}{r^2} + C$$ for some other constants B & C.

Can set $C=0$ because it simply adds a constant to the pressure.

$$p = \rho\left(\frac{2\nu^2 f}{r^2} + \frac{K}{r^2}\right)$$ where $\rho K = B$ (another const)

insert into (1 (2

$$-\frac{\nu^2}{r^3}f^2 = -\frac{\partial}{\partial r}\underbrace{\left(\frac{2\nu^2 f}{r^2}+\frac{K}{r^2}\right)}_{\frac{1}{3}P}+\frac{\nu^2}{r^3}f''$$

$$-\frac{\nu^2}{\cancel{r^3}}f(\varphi)^2 = \frac{2\left(2\nu^2 f(\varphi)+K\right)}{\cancel{r^3}}+\frac{\nu^2}{\cancel{r^3}}f''(\varphi)$$

Diff. w.r.t. φ to get rid of the constant!

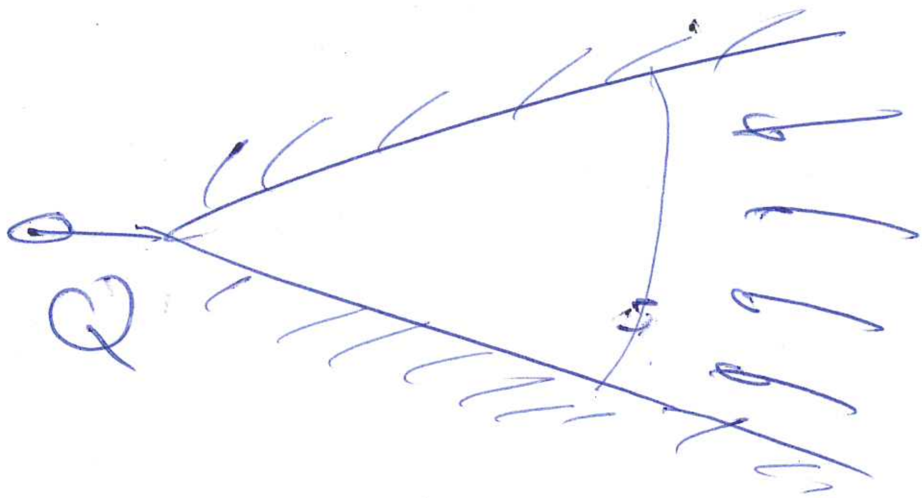
$$-\nu^2 2ff' = 4\nu^2 f' + \nu^2 f'''$$

or

$$\boxed{f''' + 4f' + 2ff' = 0}$$

3rd order nonlin. ODE for $f(\varphi)$

BCs:



$\underline{u} = \underline{0}$ at $\varphi = \pm\alpha$

$\Rightarrow f(\varphi = \alpha) = f(\varphi = -\alpha) = 0$

Inflow / outflow condition:

$$\boxed{\int_{-\alpha}^{\alpha} u r \, d\varphi = -Q}$$