$$\boxed{\begin{aligned}&\frac{\partial \hat{u}}{\partial t} = \nu \frac{\partial^2 \hat{u}}{\partial y^2}\\ &\hat{u}(y=0,t) = U \quad \text{for } t>0\\ &\hat{u} \to 0 \quad \text{as } y \to \infty\\ &\hat{u}(y,t=0) = 0\end{aligned}}$$

$\Rightarrow \hat{u} = \hat{u}(y,t;\nu,U)$

$$\boxed{\begin{aligned}&\cancel{\alpha}\frac{\partial \hat{u}}{\partial t} = \cancel{\alpha}\nu \frac{\partial^2 \hat{u}}{\partial y^2}\\ &\cancel{\alpha}\hat{u}(y=0,t) = \cancel{\alpha U} \quad \text{for } t>0\\ &\cancel{\alpha}\hat{u} \to 0 \quad \text{as } y \to \infty\\ &\cancel{\alpha}\hat{u}(y,t=0) = 0\end{aligned}}$$

$$\left.\begin{aligned}&u(y,t;\nu,\alpha U) = \alpha\,\hat{u}(y,t;\nu,U)\\ &u(y,t;\nu,U) = U\underbrace{f(y,t;\nu)}_{\text{dim less.}}\end{aligned}\right\}$$

combine y, t & ν into

$$\eta = \frac{y}{\sqrt{\nu t}}$$

This is the only nondim. combination of y, t & ν that is linear in y.

$\Rightarrow u(y, t; \nu, U) = U f(\eta)$ into PDE & BC & IC.

$$u = U f(\underbrace{y (\nu t)^{-1/2}}_{\eta})$$

$$\frac{\partial u}{\partial t} = U \underbrace{\frac{df}{d\eta}}_{f'} \frac{\partial \eta}{\partial t}$$

$$= U f' \frac{y}{\sqrt{\nu}} \left(-\frac{1}{2}\right) t^{-3/2}$$

$$\frac{\partial u}{\partial y} = U \underbrace{\frac{df}{d\eta}}_{f'} \underbrace{\frac{\partial \eta}{\partial y}}_{(\nu t)^{-1/2}}$$

$$\frac{\partial^2 u}{\partial y^2} = U f'' (\nu t)^{-1}$$

into PDE:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$-\frac{1}{2}\cancel{U}\underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta}\cancel{\frac{1}{t}} f' = \cancel{\nu}\,\cancel{U}\frac{f''}{\cancel{\nu t}}$$

$$\boxed{f'' + \frac{1}{2}\eta f' = 0} \qquad \boxed{\begin{array}{l} \eta = \frac{y}{\sqrt{\nu t}} \\ u = U f(\eta) \end{array}}$$

2nd order ODE for $f(\eta)$

BC:

$u = U$ at $y = 0$

$$\boxed{f = 1 \quad \text{for } \eta = 0}$$

$u \to 0$ as $y \to \infty$

$$\boxed{f \to 0 \quad \text{as } \eta \to \infty} \quad (*)$$

IC:

$u = 0$ as $t \to 0$

$$\boxed{f = 0 \quad \text{as} \quad \eta \to \infty}$$

same as 2^{nd} BC (*)

☺

Solve:

$f'' + \frac{1}{2}\eta f' = 0$

$F = f'$

$F' + \frac{1}{2}\eta F = 0$

$\frac{F'}{F} = -\frac{1}{2}\eta$

$\ln\left(\frac{F}{F_0}\right) = -\frac{1}{4}\eta^2$

F_0 const.

$$f' = F = F_0 \exp\left(-\frac{1}{4}\eta^2\right)$$

$$f(\eta) = A + F_0 \int_{\infty}^{\eta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

what about lower limit?
Arbitrary, here choose: ∞

$$\boxed{f(\eta) = A + \underbrace{B}_{-F_0} \int_{\eta}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi}$$

<u>BC:</u> $f \to 0$ as $\eta \to \infty$.

$$\Rightarrow A = 0.$$

$$f(\eta = 0) = 1 = B \underbrace{\int_{0}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi}_{\sqrt{\pi}}$$

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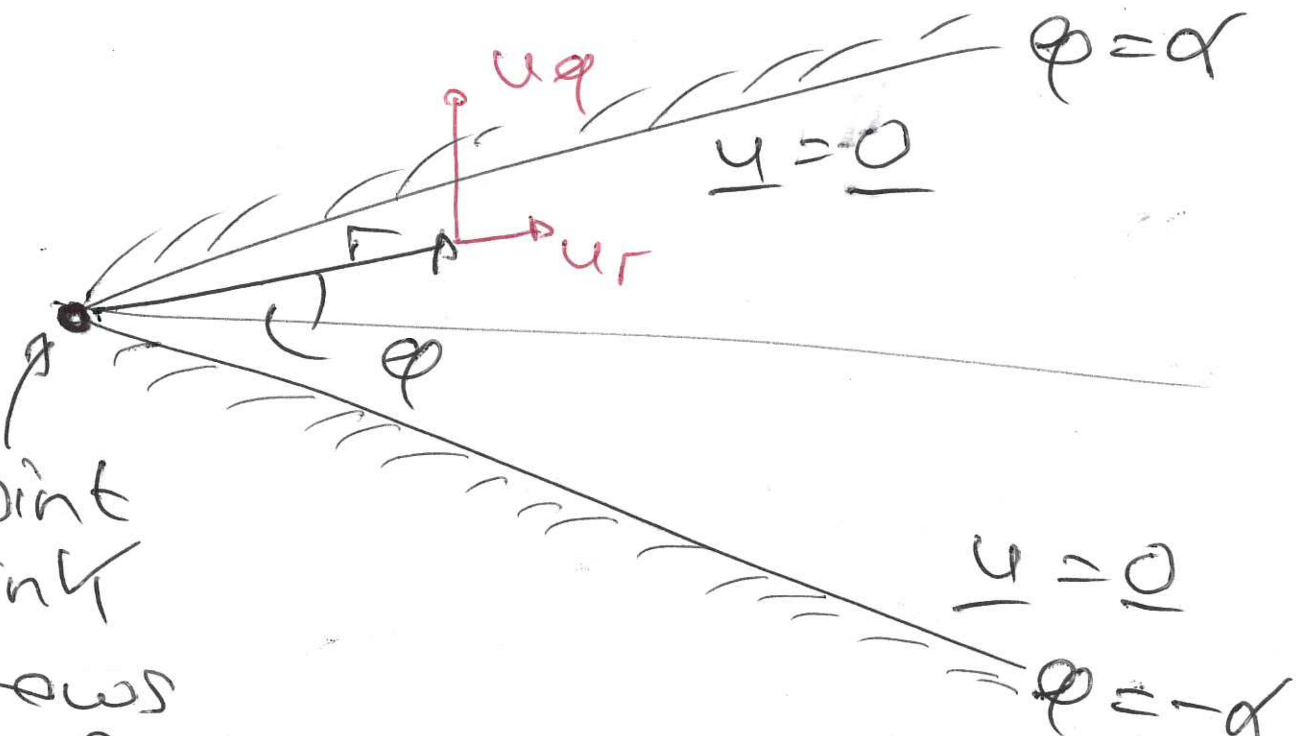
So: $u = U f(\eta)$

$$\boxed{u = \frac{U}{\sqrt{\pi}} \int_{\eta}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi}$$

where $\eta = \dfrac{y}{\sqrt{\nu t}}$

Example: Jeffrey-Hamel flow

flow in wedge



point sink draws in fluid at a rate Q $\frac{m^2}{sec}$ (syringe)

$$\underline{u} = \underline{u}(r, \varphi; \rho, \nu, Q, \alpha)$$

Dimensions:

ρ	kg/m^3
ν	m^2/sec
Q	m^2/sec
α	1
φ	1
r	m
$\underline{u}$	m/sec