- steady
- $\underline{u} = u_\varphi(r)\,\underline{e_\varphi}$ $\qquad u_\varphi = u$
- $\frac{\partial}{\partial t} = 0$
- $\frac{\partial}{\partial \varphi} = 0$
- $\nabla p = 0$
- conti ✓
- axial comp. of mom. eqn ✓
- azimuthal:

$$0 = \nu\left(\nabla^2 u - \frac{u}{r^2}\right)$$

(2

$$0 = \nu\left(\underbrace{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}}_{D^2 v(r)} - \frac{v}{r^2}\right)$$

$$\boxed{r^2 \frac{d^2 v}{dr^2} + r\frac{dv}{dr} - v = 0}$$

non-const. coeffn. 2^{nd} order linear ODE for $v(r)$.

Ansatz:

$$v(r) \sim r^\lambda$$

in ODE:

$$r^2 \lambda(\lambda-1) r^{\lambda-2} + r\lambda r^{\lambda-1} - r^\lambda \stackrel{!}{=} 0$$

$$r^\lambda \underbrace{\left(\lambda(\lambda-1) + \lambda - 1\right)}_{\stackrel{!}{=}0} \stackrel{!}{=} 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

But: check r-mom. eqn:

$$\dots - \frac{u^2}{r} = -\overset{0\ (\nabla p = 0)}{\cancel{\frac{1}{\rho}\frac{dp}{dr}}} + \dots$$

Doesn't work because our ansatz that $\nabla p = 0$ was wrong.

__Fix:__ Allow $p = p(r)$

Luckily this does not affect any of the other eqns. So can simply integrate p from

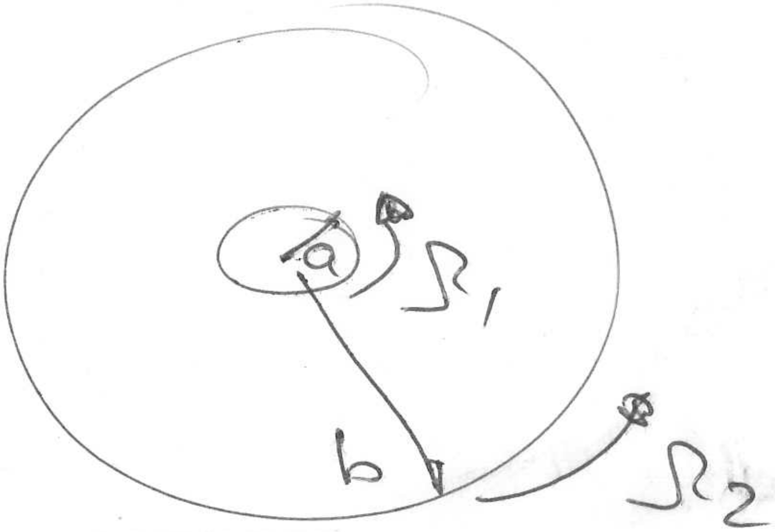
$$\frac{dp}{dr} = \rho \frac{u^2(r)}{r} \leftarrow \text{known!}$$

(centrifugal effect).

Gen. soln:

$$v(r) = A r + B \frac{1}{r}$$

BC:



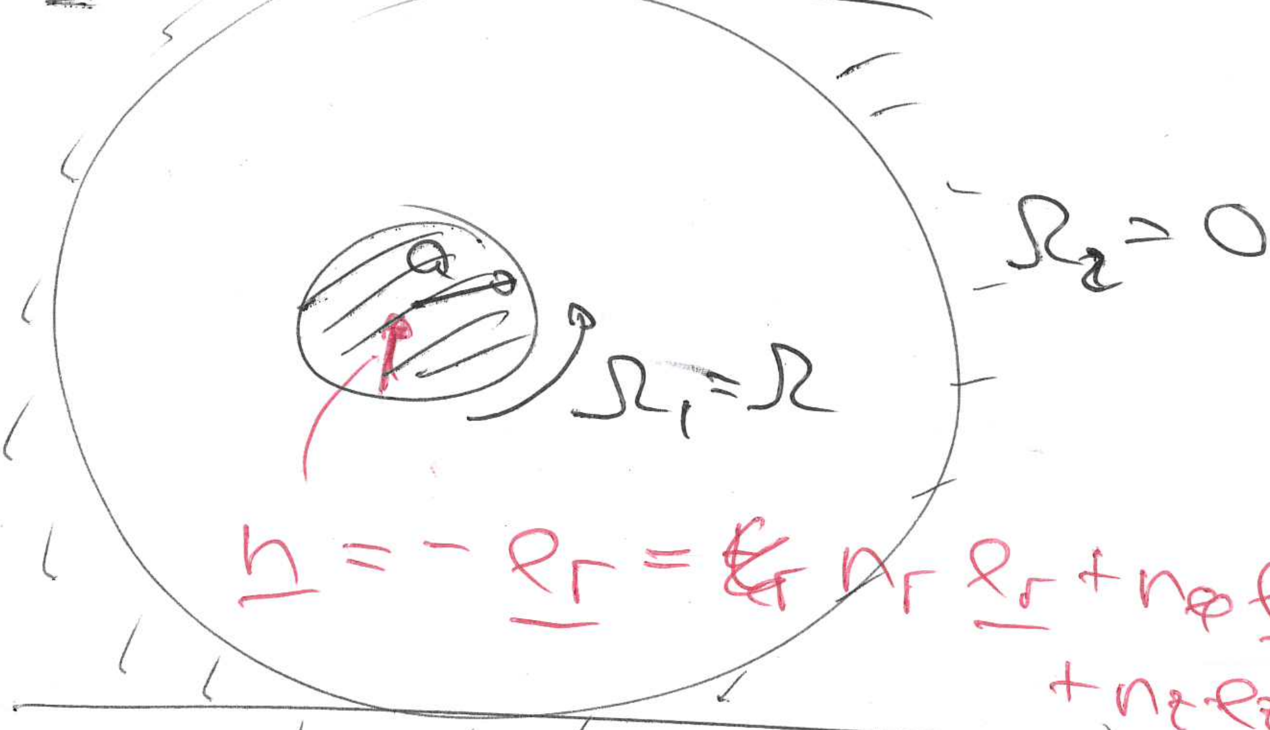
$$v(r=a) = a\Omega_1$$
$$v(r=b) = b\Omega_2$$

$$v(r) = \frac{1}{b^2-a^2}\left\{(b^2\Omega_2 - a^2\Omega_1) r - \frac{a^2 b^2(\Omega_2-\Omega_1)}{r}\right\}$$

TEST: $\Omega_1 = \Omega_2 = \Omega$

$\rightarrow v(r) = \Omega r$ ✓

Traction acting on inner cylinder.



$$\underline{n} = -\underline{e}_r = n_r \underline{e}_r + n_\varphi \underline{e}_\varphi + n_z \underline{e}_z$$

$$\boxed{v(r) = \frac{a^2 \Omega}{b^2 - a^2}\left(\frac{b^2}{r} - r\right)}$$

Traction on fluid.

$$t_i = \tau_{ij} n_j \qquad i,j = r, \varphi, z$$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu\, e_{ij}$$

Handout.

$n_r = -1$

$n_\varphi = 0$

$n_z = 0$

$u = u_r = 0 \qquad u_\varphi = v = v(r)$

$w = u_z = 0$

$$\epsilon_{rr} = \cancel{\frac{\partial u}{\partial r}} \qquad \epsilon_{\varphi\varphi} = \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{u}{r}}$$

$$\epsilon_{zz} = \cancel{\frac{\partial w}{\partial z}} \qquad \epsilon_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \cancel{\frac{1}{r}\frac{\partial u}{\partial \varphi}}\right]$$

$$\epsilon_{\varphi z} = \frac{1}{2}\left[\cancel{\frac{1}{r}\frac{\partial w}{\partial \varphi}} + \cancel{\frac{\partial v}{\partial z}}\right] \qquad \epsilon_{rz} = \frac{1}{2}\left[\cancel{\frac{\partial u}{\partial z}} + \cancel{\frac{\partial w}{\partial r}}\right]$$

From handout:

Only $$e_{r\varphi} = \frac{1}{2} r \frac{\partial}{\partial r}\left(\frac{v(r)}{r}\right)$$
is nonzero.

$$e_{r\varphi} = -\frac{a^2 b^2 \Omega}{b^2 - a^2} \frac{1}{r^2}$$

at $r = a$:

$$e_{r\varphi} = -\frac{b^2 \Omega}{b^2 - a^2}$$

$$t_i = -p n_i + 2\mu \underline{e_{ij} n_j}$$

$i = $ "r"

$$t_r = -p \underbrace{n_r}_{-1} + 2\mu\left(\cancel{e_{rr} n_r + e_{r\varphi} n_\varphi + e_{rz} n_z}\right)$$

$$\underline{\underline{t_r = p(r=a)}}$$

$i = \text{"}\varphi\text{"}$

$$t_\varphi = -p\,\cancel{n_\varphi} + 2\mu\Big(e_{\varphi r}\underbrace{n_r}_{-1} + \cancel{e_{\varphi\varphi} n_\varphi} + \cancel{e_{\varphi z} n_z}\Big)$$

$$\underline{\underline{t_\varphi}} = -2\mu\, e_{\varphi r}\Big|_{r=a} = \underline{\underline{2\mu \frac{b^2 \Omega}{b^2 - a^2}}}$$

$i = \text{"}z\text{"}$

$$\underline{\underline{t_z = 0}}$$

$$\underline{t} = \underbrace{t_r}_{p(r=a)}\,\underline{e_r} + t_\varphi\, \underline{e_\varphi}$$