$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{u}$$

$$\nabla \cdot \underline{u} = 0$$

Some eqns remain the same:

$t_i = \tau_{ij} n_j$ where i & j take values r, φ, z

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}$$

but components of e_{ij} are more complicated.

Example to illustrate the origin of some of the "new" terms:

Rigid body rotation:

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$$u = u_r(r, \varphi, z) = 0$$
$$v = u_\varphi(r, \varphi, z) = \Omega r$$
$$w = u_z(r, \varphi, z) = 0$$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} \boxed{- \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left[\cancel{\overset{\frac{\Omega}{r}}{\nabla^2 v}} - \cancel{\overset{-\frac{\Omega}{r}}{\frac{v}{r^2}}} + \cancel{\frac{2}{r^2}\frac{\partial u}{\partial \varphi}}\right], \quad \frac{\partial P}{\partial \varphi} = 0$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \cancel{\nu\nabla^2 w}, \quad \frac{\partial P}{\partial z} = 0$$

$$\operatorname{div}\mathbf{u} = \underbrace{\cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)}}_{\frac{\partial u}{\partial r} + \frac{u}{r}} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

where

$$v = \Omega r$$

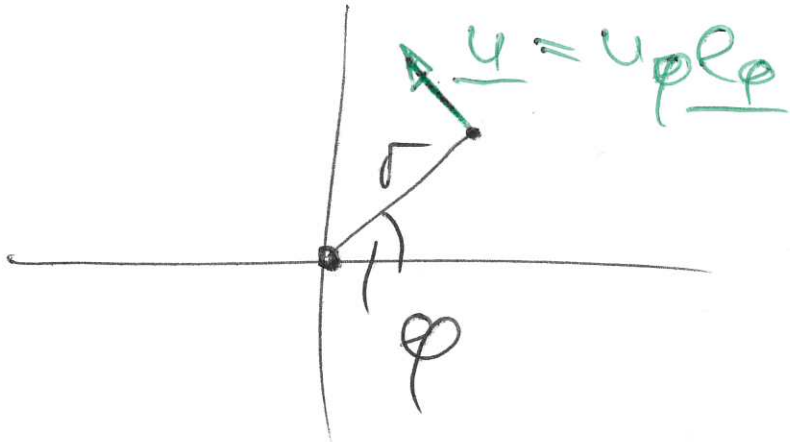
$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}} + \cancel{\frac{\partial^2}{\partial z^2}}.$$

$$\nabla^2 v = \frac{1}{r}\underbrace{\frac{\partial}{\partial r}\Big(r\underbrace{\frac{\partial(\Omega r)}{\partial r}}_{\Omega}\Big)}_{\Omega} = \frac{\Omega}{r}$$

$$\underline{u} = u_\varphi \, \underline{e_\varphi}$$

$$\downarrow$$

$$u_\varphi = \Omega r$$



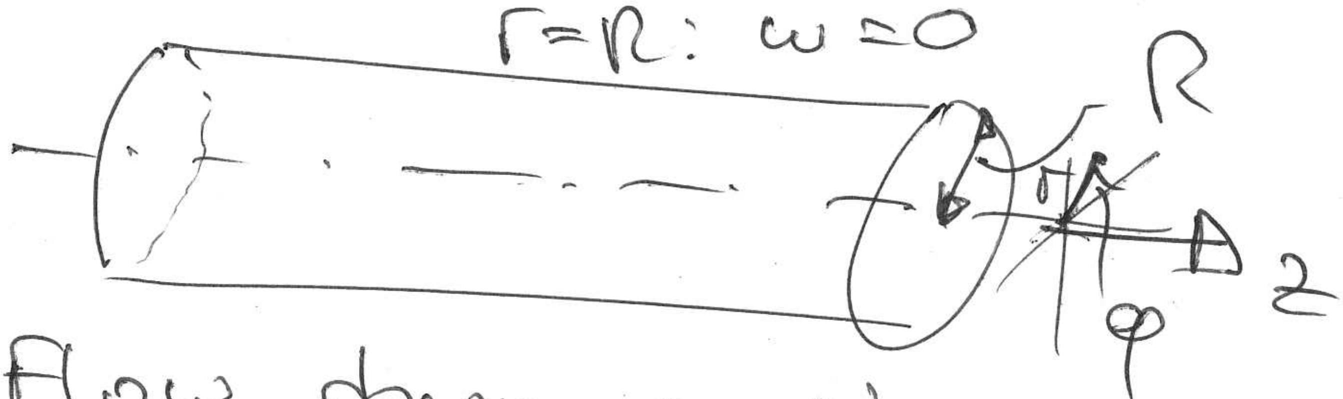
$$-\frac{u^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} = -\Omega^2 r$$

$$\boxed{p(r) = \frac{1}{2}\rho\Omega^2 r^2 + C}$$

centrifugal forces!

Example:

Hagen - Poiseuille flow



Flow down a circular pipe, driven by press. gradient G.

Assume: steady unidirectional flow in z-direction.

$$\frac{\partial}{\partial \varphi} = 0$$

$$\underline{u} = u_z \underline{e_z} = \underbrace{u_z(r)}_{w(r)} \underline{e_z}$$

$$p = p(r, z)$$

Into N. St. eqns:

<u>r - comp:</u> $0 = -\frac{1}{\rho}\frac{\partial p}{\partial r}$ (EXERCISE)

$$p = p(z)$$

<u>φ - comp:</u> $0 = 0$

<u>continuity:</u> $0 = 0$

<u>z - comp:</u>

$$\dots + w\cancel{\frac{\partial w}{\partial z}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{fct of } z} + \underbrace{\nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)}_{\text{fct of } r}$$

$\Downarrow$

constant

so $\frac{\partial p}{\partial z} = G$

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$$\frac{G}{\rho} = \underbrace{\nu}_{\frac{\mu}{\rho}} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right)$$

$$\underbrace{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}} = \frac{G}{\mu}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r} \right) = \frac{G}{\mu}$$

integrate twice

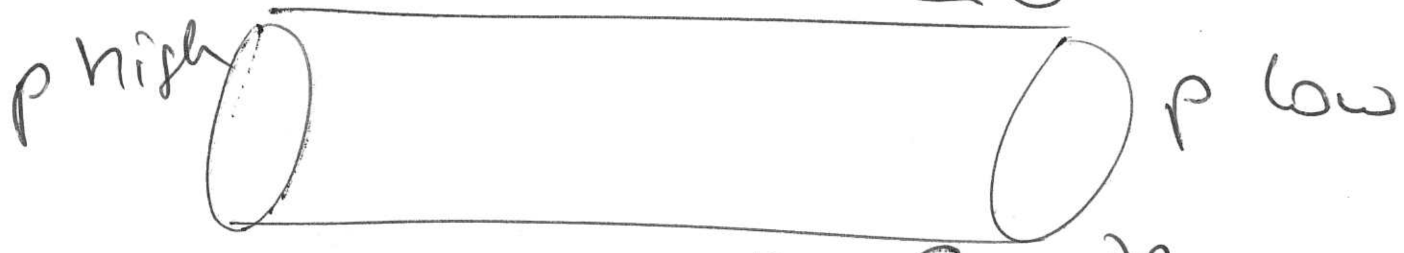
$$w(r) = \frac{1}{4}\frac{G}{\mu} r^2 + A \ln r + B$$

BC: $w(r = R) = 0$ (no slip)

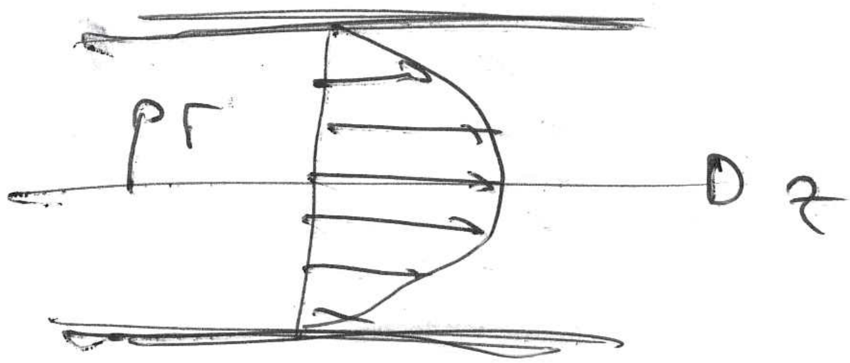
solution finite at $r = 0$:

$A = 0$

$$w(r) = \frac{1}{4}\frac{G}{\mu}\underbrace{(r^2 - R^2)}_{<0}$$

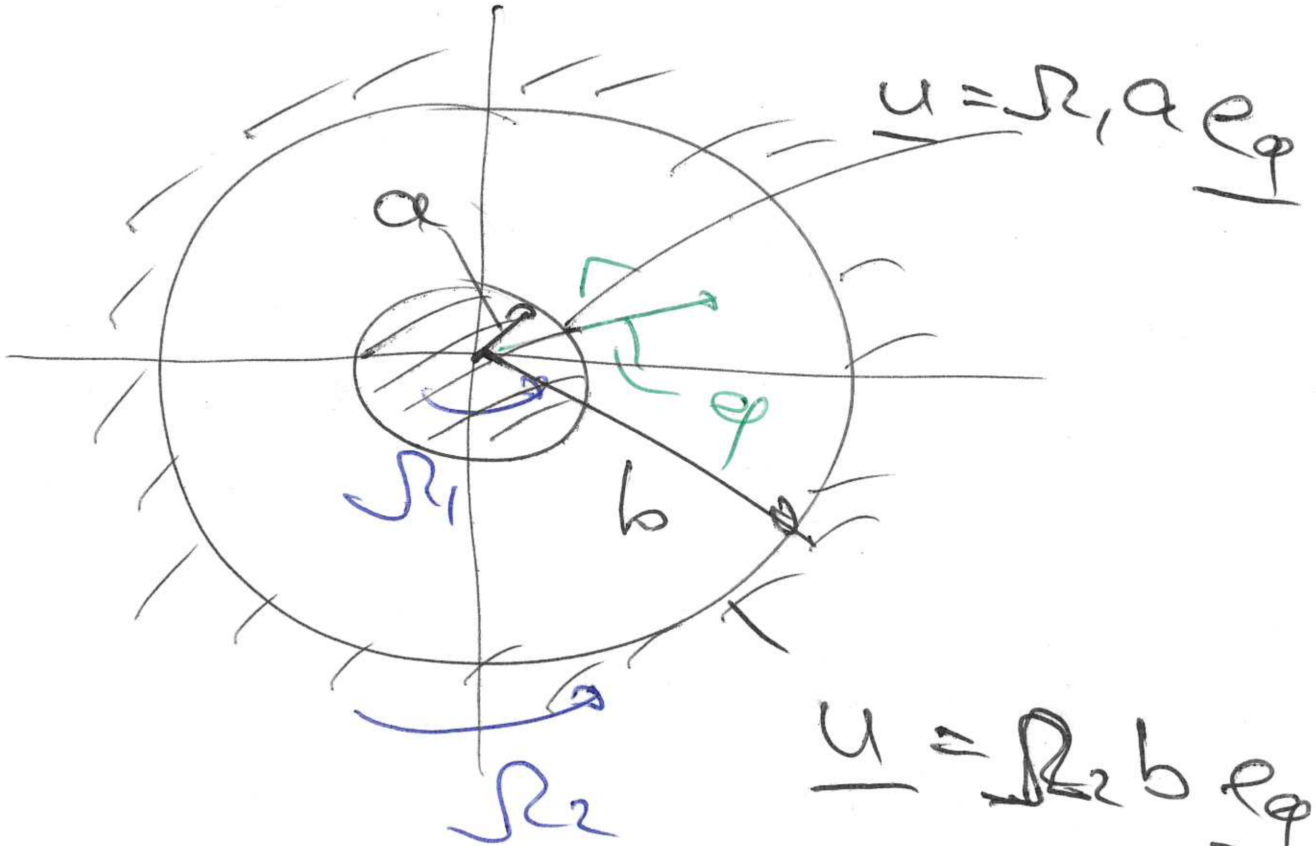


$$\Rightarrow \quad G = \frac{\partial p}{\partial z} < 0$$



Example:

Circular Couette flow



Assumptions:

- steady $\frac{\partial}{\partial t} = 0$
- $\underline{u} = u_\varphi \underline{e_\varphi}$
- $\frac{\partial}{\partial \varphi} = 0$
- $\frac{\partial}{\partial z} = 0$

$\Big\}\quad \underline{u} = u_\varphi(r)\, \underline{e_\varphi}$

Flow is driven by wall motion not by pressure: $\nabla p = \underline{0}$.

into eqns:

$$\nabla^2 u - \frac{u}{r^2} = 0$$

from azimuthal momentum eqns.

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$u=0 \quad w=0 \quad v(r) \quad \nabla p = 0$

$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial z} = \frac{\partial}{\partial t} = 0$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \varphi} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right],$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = \cancel{-\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \boxed{\nu\left[\nabla^2 v - \frac{v}{r^2}\right.} + \cancel{\left.\frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right]},$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = \cancel{-\frac{1}{\rho}\frac{\partial P}{\partial z}} + \cancel{\nu\nabla^2 w},$$

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$