Example: The vibrating plate:

No body force.

flow driven by wall motion
(~ Couette flow)

expect parallel flow in x-direct.
$\frac{\partial p}{\partial x} = G = 0$; but time-dep.

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}} \qquad u = u(y,t)$$

BC. $u = U\cos\omega t$ @ $y = 0$

$u \to 0$ as $y \to \infty$

Look for periodic steady state soln.

$$u(y,t) = f(y)\cos(\omega t + \phi(y))$$

easier (as always) using complex vars.

$$u(y,t) = f(y)\, e^{i\omega t}$$

into $\quad \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2}$

$$i\omega f = \nu f''$$

$$f'' - \frac{i\omega}{\nu} f = 0$$

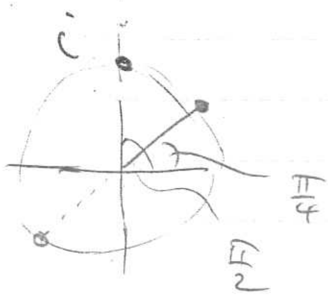
linear ODE with const. coeffs

$$f \sim e^{\lambda y}$$

$$\lambda^2 - \frac{i\omega}{\nu} = 0$$

$$\lambda = \pm \sqrt{\frac{i\omega}{\nu}}$$

$$\sqrt{i} = \tfrac{1}{2}\sqrt{2}\,(1+i)$$



$$\lambda = \pm (1+i) \sqrt{\frac{\omega}{2\nu}}$$

(neg. root is already in $\pm$)

$$f(y) = A e^{(1+i)\sqrt{\frac{\omega}{2\nu}}\,y} + B e^{-(1+i)\sqrt{\frac{\omega}{2\nu}}\,y}$$

BC:

$$f(0) = U = A + B$$

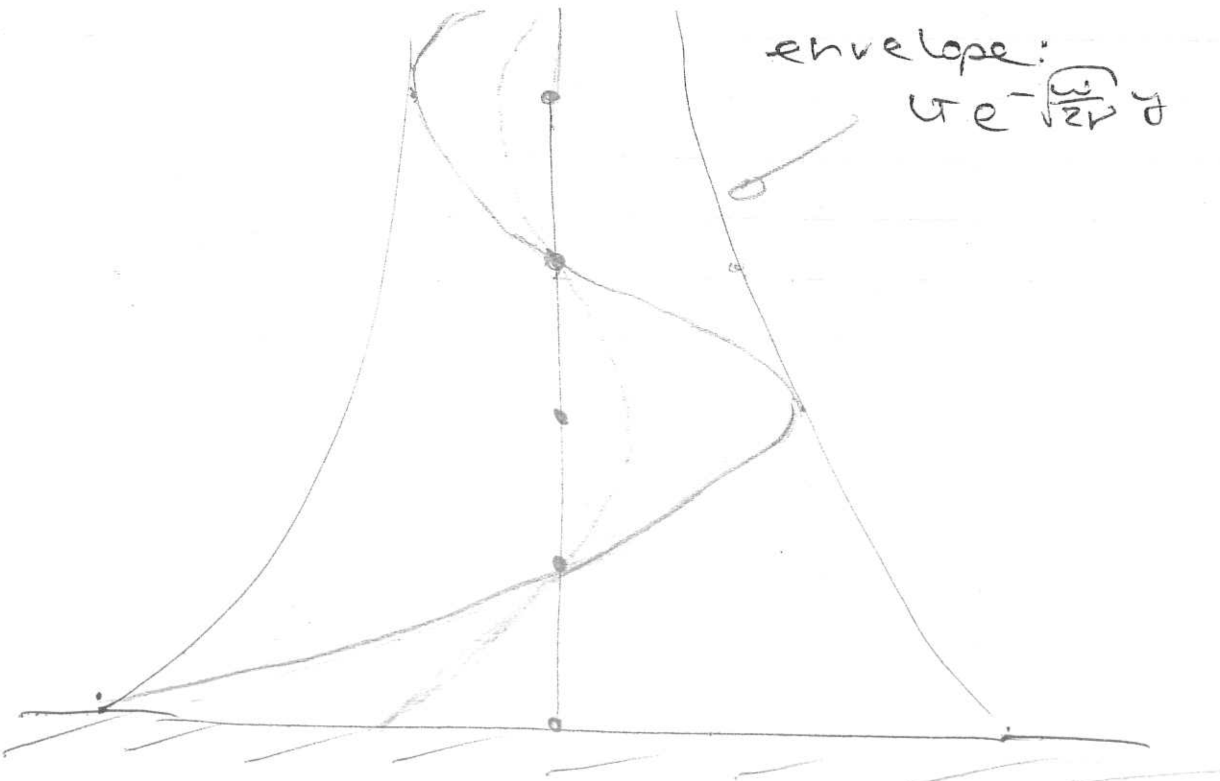
$$f \to 0 \text{ as } y \to +\infty : \quad A = 0$$

$$B = U$$

$$u(y,t) = U e^{-(1+i)\sqrt{\frac{\omega}{2\nu}}\,y} e^{i\omega t}$$

convert back to real:

$$\boxed{u(y,t) = U e^{-\sqrt{\frac{\omega}{2\nu}}\,y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\,y\right)}$$



Note:

- the higher the frequency the more rapidly the velocity perturbation decays
- the larger the viscosity the more slowly it decays.

Makes sense:

- Viscosity would like the entire volume of fluid to move with the wall
- this is resisted by the fluid's inertia

$$\delta = \sqrt{\frac{\omega}{2\nu}} = \sqrt{\frac{\omega\rho}{2\mu}}$$

Rapid oscill or "heavy" fluid lead to large inertial forces.

∃ Fluid would like to stay @ rest.

Remark: In all these cases we have assumed that the flow is parallel. This assumption simplified the nonlin. N.St. eqns to linear eqns which we could solve by elementary methods.
(The boundary cond. were consistent with our assumptions).

$\exists$ we have found a soln. to the full non. lin. N.St. eqns.

The nonlinearity of these eqns often allows other solns to exist ~~[illegible]~~ (i.e. under the same BC).

$\to$ Stability.

Curvilinear coords

So far: derived eqns in cartesian coords & used index notation as a shorthand for components w.r.t. cartesian basis vectors

$$\underline{u} = u_1 \underline{e_1} + u_2 \underline{e_2} + u_3 \underline{e_3}$$

$$\rightarrow u_i$$

Eqns can be transformed to different coord. system

E.g.: $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned} \quad \rightarrow \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \varphi &= \mathrm{atan}\left(\frac{y}{x}\right) \end{aligned}$$

$$\begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial\varphi}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial\varphi}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial\phi}{\partial r} \\ \frac{\partial\phi}{\partial\varphi} \end{pmatrix}$$

& transform derivs.

$$? \quad \nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}$$

This is for a scalar!

N.St. eqns contain vector quantities:

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{u}$$

$$\nabla \cdot \underline{u} = 0$$

must transform vector components as well!

$$\underline{u} = u_1 \underline{e_1} + u_2 \underline{e_2} + u_3 \underline{e_3}$$

$$= u_r \underline{e_r} + u_\varphi \underline{e_\varphi} + u_z \underline{e_z}$$

(in cyl. polars)

Now: orientation of basis vectors depends on the coords!

$$\underline{e_r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

& differential operators also act on basis vectors

A MESS!

see handout

$$\boxed{\text{Can still use index notation! e.g. } \quad t_i = \tau_{ij} n_j \qquad i,j = [r, \varphi, z] \text{ e.g.}}$$