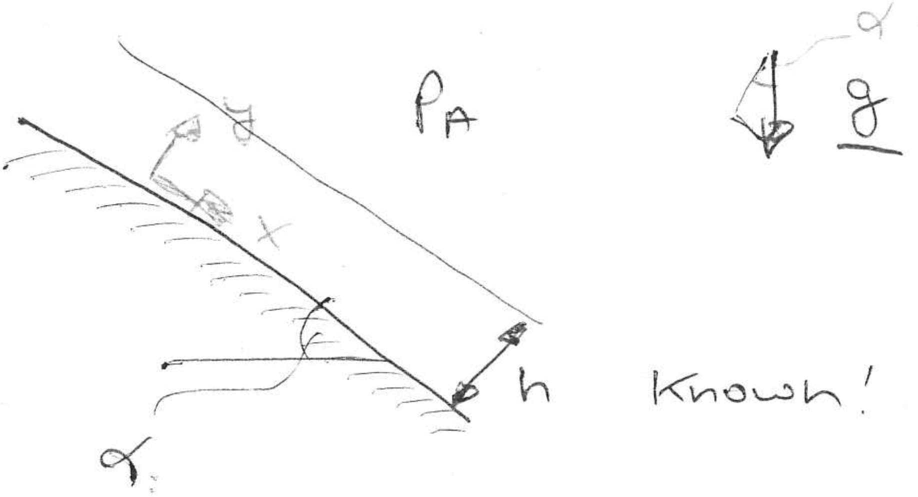
Example: Flow down an inclined plane



Decompose gravitational body force into x & y directions:

$$\underline{F} = \underline{g} = g \sin\alpha \, \underline{e_x} - g \cos\alpha \, \underline{e_y}$$

Again: Assume parallel flow & steady & indep. of z.

So use parallel flow eqns with body force:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + g\rho \sin\alpha + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha} \quad (1)$$

y-comp:

$$0 = -\frac{\partial p}{\partial y} - g\rho\cos\alpha \quad (2)$$

z-comp:

$$0 = -\frac{\partial p}{\partial z} \longrightarrow p = p(x,y)$$

B.C. No slip @ y=0

$$u = 0 \quad @ \; y = 0$$

y=h: free surface: Air is far less viscous than fluid

⇒ surface traction is given by the air pressure:

$$\underline{t} = -p_A \, \underline{e}_y \qquad \text{(applied traction!)}$$

$$\underline{n} = \underline{e}_y \quad \text{(outer normal)}$$

$t_2 = -p_A$ $\qquad n_2 = 1$

$t_1 = 0$ $\qquad n_1 = n_3 = 0$

$t_3 = 0$

$$\left.\begin{aligned} t_i &= \tau_{ij} n_j \\ t_i &= \left[-p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] n_j \end{aligned}\right\} @ \; y = h$$

$$t_i = -p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

for all 3 free indices $i = 1, .., 3$

$i = 2$:

$$t_2 = -p_A = -p n_2 + \mu\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) n_2$$

all other terms in summation over j vanish since $n_2 = n_3 = 0$

$$\frac{\partial u_2}{\partial x_2} = \frac{\partial v}{\partial y} = 0$$

since ~~$u = u(y)$~~ $v = 0$)

$\Rightarrow$

$$\boxed{p = p_A \quad @ \; y = h}$$

$i=2$

$n_2 = 0$ $\quad u_2 = 0$

$$t_2 = 0 = -p n_1 + \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) n_2$$

$$0 = \mu \frac{\partial u_1}{\partial x_2}$$

$$\boxed{0 = \mu \frac{\partial u}{\partial y} \quad @ \; y = h}$$

∃ no tangential shear stress

$i=3$

$n_3 = 0$ $\quad u_3 = 0$ $\quad u = u(y)$

$$t_3 = 0 = -p n_3 + \mu \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) n_2$$

$$0 = 0.$$

(consistent)

$\Rightarrow$ integrate (2) w.r.t. y

$$\frac{\partial p}{\partial y} = -g\rho\cos\alpha$$

$$p(x,y) = -g\rho\cos\alpha\, y + f(x)$$

(partial deriv!)

Now pressure B.C. @ $y = h$

$$p_A = -\rho g\cos\alpha\, h + f(x) \qquad \forall x$$

$$f(x) = \text{const.} = p_A + \rho g h\cos\alpha$$

$$\Rightarrow \boxed{p = p_A + \rho g\cos\alpha(h-y)}$$

- Note:
  - pressure increases through the film thickness due to hydrostatic effect.
  - pressure is indep. of x: flow is entirely driven by gravity.

Now (1):

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin\alpha + \mu \frac{\partial^2 u}{\partial y^2}$$

$\rho g \sin\alpha$ acts like a driving applied pressure gradient.

$$u = -\frac{1}{2}\frac{\rho g \sin\alpha}{\mu} y^2 + Ay + B$$

$$u(0) = 0 = B$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=h} = 0 = -\frac{g\sin\alpha}{\nu}h + A$$

$$A = +\frac{g\sin\alpha}{\nu}h$$

$$\boxed{u = \frac{g\sin\alpha}{\nu}\left[hy - \frac{1}{2}y^2\right]}$$