Index notation

Various ways of expressing a vector:

$$\underset{\uparrow}{\underline{a}} = \underbrace{a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

symbolic

in components relative to the basis $(\underline{e}_1, \underline{e}_2, \underline{e}_3) = (\underline{i}, \underline{j}, \underline{k})$

Convention 1:

Simply write down one generic term of each vector eqn:

$$\underline{c} = \underline{a} + \underline{b} \Rightarrow c_i = a_i + b_i$$

i is a free index & takes values 1, 2, 3.

Example:

$$\nabla \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix} \rightarrow \frac{\partial \phi}{\partial x_i}$$

Convention 2:

Summation convention.

Rule: Automatically sum over any repeated indices/ dummy indices

Example:

$$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$= \cancel{\sum_{j=1}^{3}} \underbrace{u_j v_j} = \underbrace{u_j v_j}$$

$$= \underbrace{u_k v_k}$$

Repeated indices can be renamed arbitrarily. They are called dummy indices.

$$\left.\begin{array}{l} \operatorname{div} \underline{u} = \\ \nabla \cdot \underline{u} = \end{array}\right\} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \cancel{\sum_{j=1}^{3}} \frac{\partial u_j}{\partial x_j} = \frac{\partial u_k}{\partial x_k}$$

Higher order "tensors"

So far: used index notation ~~vec~~ for vectors (objects with one free index in each component)

Higher order tensors arise naturally in many applications

E.g.:

$$\underline{\sigma} = \underset{\sim}{T} \cdot \underline{n}$$

unit vector

stress vector

stress tensor (3x3 matrix)

$$\sigma_i = T_{ij} n_j$$

(matrix vector product)

<u>Note:</u> Every term in an eqn. in index notation has to have the same free index (here: i)

A special second order tensor is the

<u>Kronecker delta</u>

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$[\delta_{ij}] = \begin{array}{c|c} & j \rightarrow \\ \hline i \downarrow & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

= unit matrix.

δ_{ij} has an interesting property in summations:

$$b_i = \boxed{\delta_{ij} a_j = a_i}$$

"multiplication by δ_{ij} changes indices"