# Chapter 8

# The Streamfunction and Vorticity

- For 2D incompressible flows, it is possible to recast the Navier-Stokes equations in an alternative form in terms of the streamfunction and the vorticity.
- In many applications, the streamfunction-vorticity form of the Navier Stokes equations provides better insight into the physical mechanisms driving the flow than the 'primitive variable' formulation in terms of $u, v$ and $p$.
- The streamfunction and vorticity formulation is also useful for numerical work since it avoids some problems resulting from the discretisation of the continuity equation.

| Unless specifically stated, all results in this chapter are restricted to 2D incompressible flows. |
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## 8.1 The Streamfunction

- The streamfunction is defined as
$$\psi_A(P) = \int_A^P \mathbf{u} \cdot \mathbf{n} \, ds, \tag{8.1}$$
where the integral has to be evaluated along a line from the arbitrary but fixed point A to point P. $\mathbf{n}$ is the unit normal on the line from A to P. We regard $\psi_A(P)$ as a function of the location of point P.

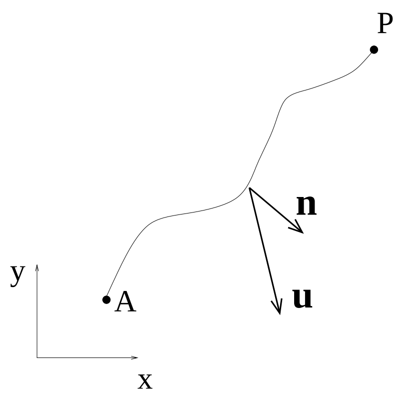


Figure 8.1: Sketch illustrating the definition of the streamfunction.

- The sketch in Fig. 8.1 shows that $\mathbf{u} \cdot \mathbf{n}$ is equal to the component of the velocity $\mathbf{u}$ that crosses the line AP. Therefore $\psi_A(P)$ represents the volume flux (per unit depth in the $z$-direction) through the line between A and P.

- Evaluating $\psi_A(P)$ along two different paths and invoking the integral form of the incompressibility constraint shows that $\psi_A(P)$ is path-independent, i.e. its value only depends on the locations of the points A and P.
- Changing the position of point A only changes $\psi_A(P)$ by a constant. It turns out that for all applications such changes are irrelevant. It is therefore common to suppress the explicit reference to A. Hence, we regard $\psi_A(P)$ as a function of the spatial coordinates only, i.e. $\psi_A(P) = \psi(P) = \psi(x, y)$.
- Streamlines are lines which are everywhere tangential to the velocity field, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal to the streamline. Hence the streamfunction $\psi$ is constant along streamlines.
- Note that stationary impermeable boundaries are also characterised by $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal on the boundary. Therefore, $\psi$ is also constant along such boundaries.
- Invoking the integral incompressibility constraint for an infinitesimally small triangle shows that $\psi$ is related to the two cartesian velocity components $u$ and $v$ via

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{8.2}$$

- Similarly, in plane cylindrical polars, the velocity components are given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad \text{and} \quad u_\varphi = -\frac{\partial \psi}{\partial r}. \tag{8.3}$$

- Flows which are specified by a streamfunction automatically satisfy the continuity equation since

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0. \tag{8.4}$$

- For 2D flows, the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ only has one non-zero component (in the $z$-direction), i.e. $\boldsymbol{\omega} = \omega \mathbf{e}_z$ where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8.5}$$

Using the definition of the velocities in terms of the streamfunction shows that

$$\omega = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) \tag{8.6}$$

and therefore

$$\omega = -\nabla^2 \psi, \tag{8.7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2D Laplace operator.

## 8.2 The Streamfunction-Vorticity form of the Navier-Stokes equations

- Straightforward algebraic manipulation of the 3D momentum equations transforms them into the *vorticity transport equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega} \tag{8.8}$$

(see the separate handout for the derivation; this equation is valid in 3D).

- This equation shows that the rate of change of the vorticity of material particles, $D\boldsymbol{\omega}/Dt$, is controlled by 'vortex stretching' (described by $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$; this is a familiar result from inviscid fluid mechanics) and by diffusion (described by $\nu \nabla^2 \boldsymbol{\omega}$). The diffusion of vorticity only occurs in viscous flows.

- For 2D flows, vortex stretching is absent since $\mathbf{u} = u(x,y)\ \mathbf{e}_x + v(x,y)\ \mathbf{e}_y$ and $\boldsymbol{\omega} = \omega(x,y)\ \mathbf{e}_z$ and therefore $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = 0$.

- For 2D flows, the scalar vorticity transport equation

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega \tag{8.9}$$

together with the equation for the vorticity in terms of the streamfunction

$$\omega = -\nabla^2\psi \tag{8.10}$$

and

$$u = \partial\psi/\partial y \quad \text{and} \quad v = -\partial\psi/\partial x \tag{8.11}$$

provide the streamfunction-vorticity formulation of the Navier-Stokes equations. It consists of only two PDEs for the scalars $\omega$ and $\psi$ rather than the three PDEs for $u, v$ and $p$ in the 'primitive variable' form.

- In the limit of zero Reynolds number, only one fourth-order PDE for the streamfunction $\psi$ needs to be solved, namely the biharmonic equation

$$\nabla^4\psi = 0, \tag{8.12}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{8.13}$$

This can be shown by, e.g., taking the curl of the Stokes equations.

### 8.2.1 The streamfunction and the biharmonic equation in cylindrical polars

- In cylindrical polars, $(r, \varphi)$ the relation between the streamfunction $\psi(r, \varphi)$ and the velocity components $u_r$ and $u_\varphi$ is:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \tag{8.14}$$

and

$$u_\varphi = -\frac{\partial \psi}{\partial r}, \tag{8.15}$$

where $\mathbf{u} = u_r \mathbf{e}_r + u_\varphi \mathbf{e}_\varphi$.

- The biharmonic equation in polar coordinates:

$$\nabla^4 \psi(r, \varphi) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right]\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi^2}\right] \tag{8.16}$$

$$\nabla^4 \psi(r, \varphi) = \psi_{,rrrr} + \frac{2}{r}\psi_{,rrr} - \frac{1}{r^2}(\psi_{,rr} - 2\psi_{,rr\varphi\varphi}) + \frac{1}{r^3}(\psi_{,r} - 2\psi_{,r\varphi\varphi}) + \frac{1}{r^4}(4\psi_{,\varphi\varphi} + 2\psi_{,\varphi\varphi\varphi\varphi}) \tag{8.17}$$

- For axisymmetric solutions:

$$\nabla^4 \psi(r) = \frac{1}{r}\left[r\left(\frac{1}{r}[r\psi_{,r}]_{,r}\right)_{,r}\right]_{,r} \tag{8.18}$$

$$\nabla^4 \psi(r) = \psi_{,rrrr} + \frac{2}{r}\psi_{,rrr} - \frac{1}{r^2}\psi_{,rr} + \frac{1}{r^3}\psi_{,r} \tag{8.19}$$

- The general form of the solution of the biharmonic equation in cylindrical polars is known. It can be represented by superposition of the following solutions:
  - The general axisymmetric solution:

$$\psi(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \tag{8.20}$$

  - The general separated non-axisymmetric solution:

    **For $n = 1$:**

$$\psi(r, \varphi) = \left(Ar + \frac{B}{r} + Cr^3 + Dr \ln r\right) \quad \cos(\varphi) + \left(ar + \frac{b}{r} + cr^3 + dr \ln r\right) \quad \sin(\varphi) \tag{8.21}$$

    **For $n \geq 2$:**

$$\psi(r, \varphi) = \sum_{n=2}^{\infty} \left(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}\right) \quad \cos(n\varphi) + \left(a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}\right) \quad \sin(n\varphi) \tag{8.22}$$

The coefficients $(A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1, a_1, b_1, c_1, d_1, A_2, B_2, C_2, D_2, a_2, b_2, c_2, d_2, ...)$ have to be determined from the boundary conditions.