1

$$\boxed{\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=\rho f_i-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j^2} \qquad \frac{\partial u_j}{\partial x_j}=0}$$

IC: $\underline{u}(t=0)=\underline{u}_0$

BC: $\underline{u}=\underline{V}$ (given)

or free surface

Two conditions

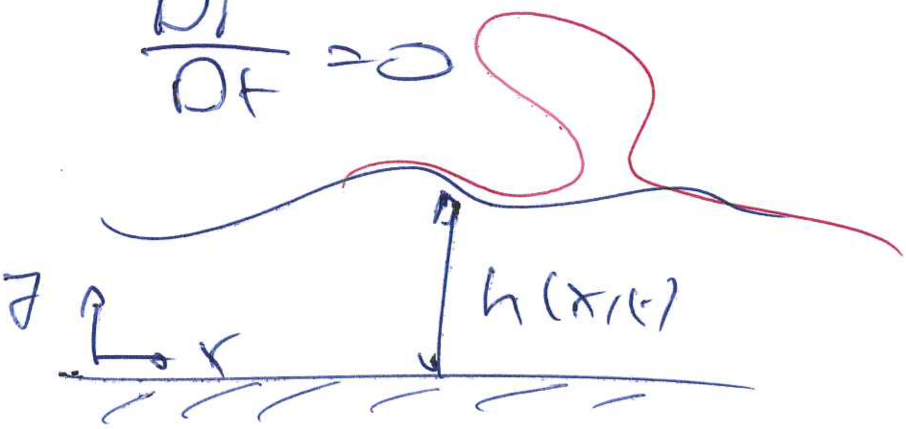
(a) kinematic

(b) traction

Kinematic BC

$F(x,y,z,t)=0$ defines surface

$$\frac{DF}{Dt}=0$$



$$F=F(x,y,t)=h(x,t)-y$$

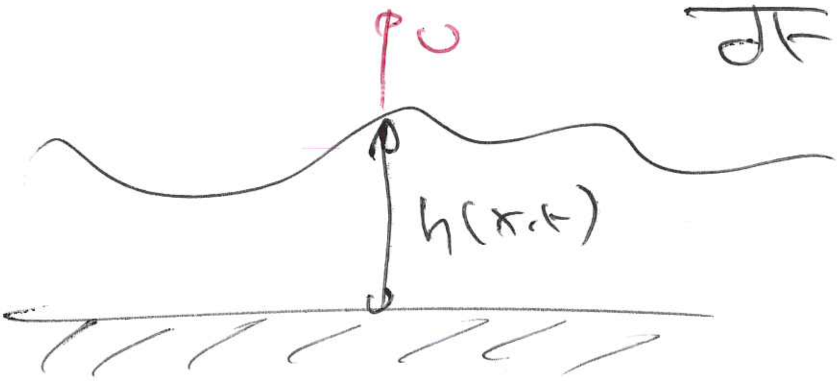
so

$$\frac{Df}{Dt} = \boxed{\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - v = 0}$$

Special cases:

(1) Only vertical veloc:

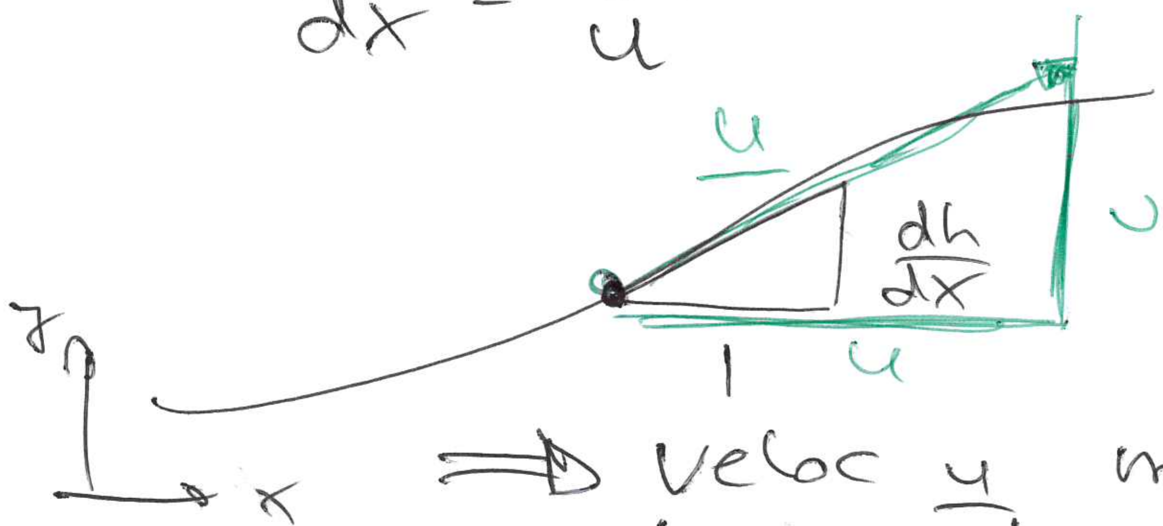
$u = 0$: $\quad \frac{\partial h}{\partial t} = v$



(2) fixed free surface posn

$$\frac{\partial h}{\partial t} = 0: \quad u\frac{\partial h}{\partial x} = v$$

$$\frac{dh}{dx} = \frac{v}{u}$$

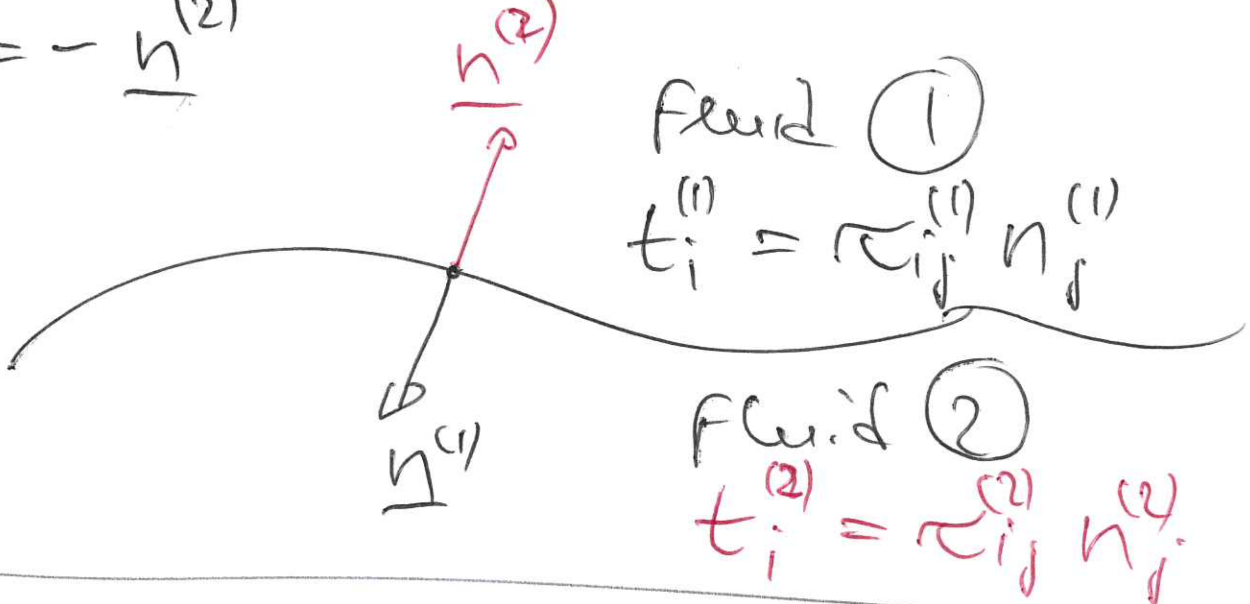


$\Rightarrow$ veloc $\underline{u}$ must be tangent to surface

(b) Traction BC.

Stress must be continuous across the free surface (apart from surface tension)

$\underline{n}^{(1)} = -\underline{n}^{(2)}$



$$\boxed{t_i = \tau_{ij}\, n_j}$$

t_i: traction applied to fluid

τ_{ij}: stress tensor in that fluid

n_j: outer unit normal onto fluid

Action = Reaction

$$\underline{t}^{(1)} = -\underline{t}^{(2)}$$

$$\boxed{\tau^{(1)}_{ij} n_j = \tau^{(2)}_{ij} n_j}$$

where n_j is either $n_j^{(1)}$ or $n_j^{(2)}$.

i.e. for any unit normal to surface.

<u>Example:</u>

Hydrostatics (or inviscid fluid, $\mu = 0$)

$$\tau_{ij} = -p\,\delta_{ij}$$

$$-p^{(1)} \underbrace{\delta_{ij} n_j} = -p^{(2)} \underbrace{\delta_{ij} n_j}$$

$$-p^{(1)} \underline{n} = -p^{(2)} \underline{n}$$

$$\underline{n}\left(p^{(1)} - p^{(2)}\right) = \underline{0}$$

$$p^{(1)} = p^{(2)}.$$

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Modification in the presence of surface tension:

$$\tau_{ij}^{(1)} n_j + \sigma \kappa n_i = \tau_{ij}^{(2)} n_j$$

surface tension

mean curvature of surface.

§... Parallel flows

N.St. are extremely complicated because of the nonlin terms $u_j \frac{\partial u_i}{\partial x_j}$.

These terms vanish for certain flows. E.g. parallel flows:

Assume flow is unidirectional

w.l.o.g. assume:

$$\boxed{\begin{array}{l} u = u_1(x, y, z, t) \\ v = u_2 = w = u_3 = 0 \end{array}}$$

Consequences?

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y, z, t)$$

$u = u(y, z, t)$

$v = 0$

$w = 0$

$\nu = \frac{\mu}{\rho}$

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{w\frac{\partial v}{\partial z}} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{v\frac{\partial w}{\partial y}} + \cancel{w\frac{\partial w}{\partial z}} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\cancel{\frac{\partial^2 w}{\partial x^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}}\right)$$

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

(1) : $\rho \frac{\partial u}{\partial t} = \rho f_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$

(2) : $\rho f_y = \frac{\partial p}{\partial y}$

(3) : $\rho f_z = \frac{\partial p}{\partial z}$

These eqns are linear!