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(incompressible) Newtonian fluids

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

$$= -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

pressure. viscosity

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insert into Cauchy's eqn:

$$\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$= \rho f_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

$$= \rho f_i + \left(-\frac{\partial p}{\partial x_i} + \mu \underbrace{\frac{\partial^2 u_i}{\partial x_j^2}}_{\nabla^2 u_i} + \cancel{\mu \frac{\partial}{\partial x_i} \underbrace{\left(\frac{\partial u_j}{\partial x_j}\right)}_{\mathrm{div}\,\underline{u}}}\right)$$

$$\boxed{\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho F_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}}$$

+ eqn. of continuity:

$$\boxed{\frac{\partial u_j}{\partial x_j} = 0}$$

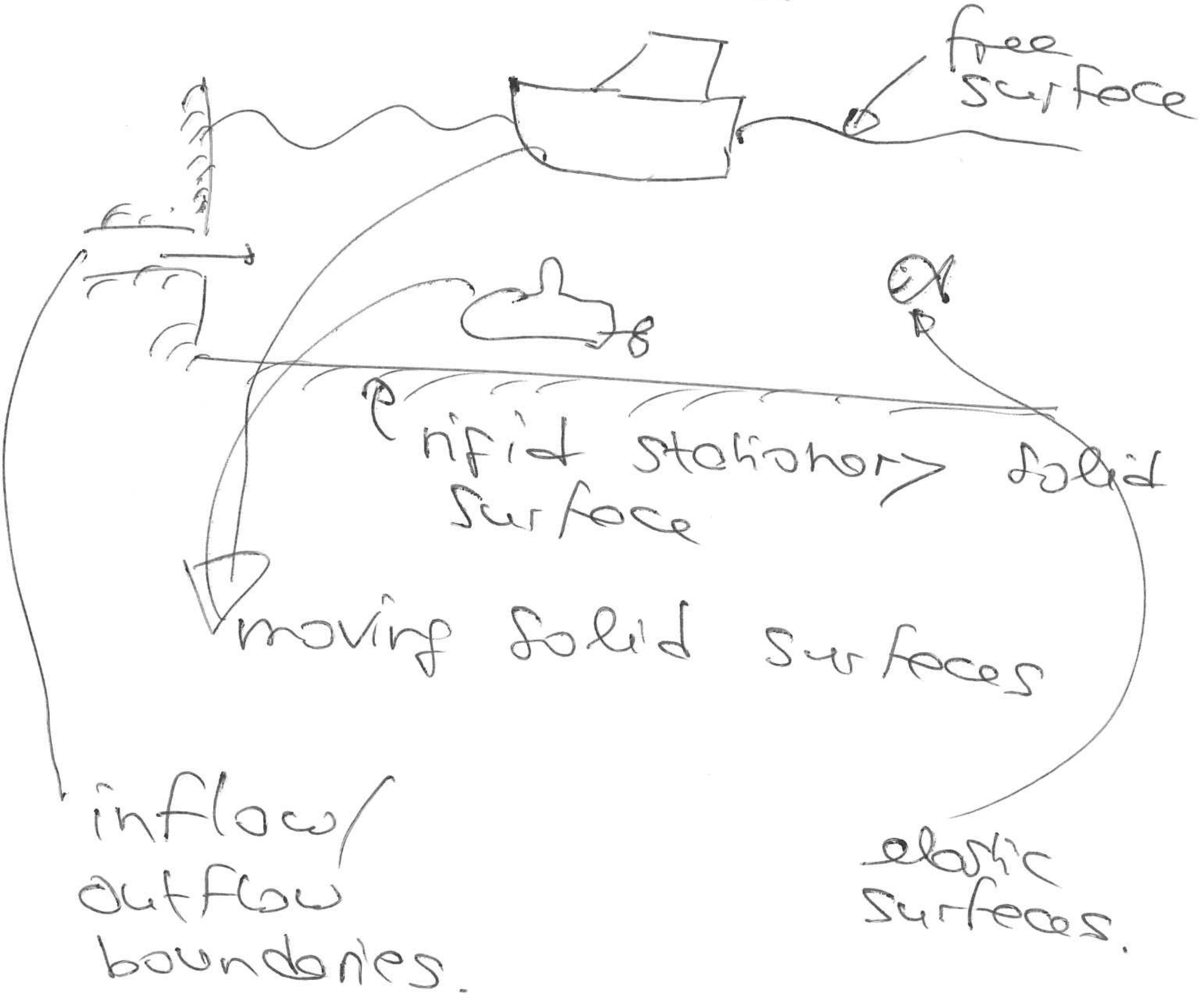
These are the famous Navier Stokes eqns.

momentum eqns.

They are a system of four coupled, nonlinear, second order (in space) ~~eqns~~ PDEs for 3 veloc components u_i & the pressure p.

Boundary & initial conditions

General problem



Initial conditions:

Need to specify $u_i(x_j, t=0)$ for time dep. problems.

Note: No IC for pressure.

Boundary conditions

(i) in/outflow BCs

$u_i = v_i$ (prescribed)

(ii) on all solid surfaces

"No slip & no penetration"

Solid veloc = fluid veloc.

(given)

$u_i = v_i$

solid veloc (given)

Special case: stationary wall:

$u_i = 0$

(iii) on free surfaces

we need 2 conditions:

- Kinematic BC
- traction BC

(a) Kinematic BC

The posn of the free surface can always be described implicitly as

$$F(x, y, z, t) = 0$$

or in 2D

$$F(x, y, t) = 0$$

At ~~least~~ least locally this can be inverted to

$$z = h(x, y, t)$$

or

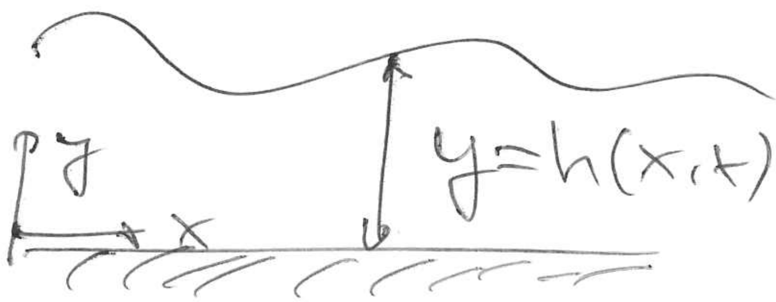
$$y = h(x, t)$$

Physical observation:

Fluid particles on free surface stay on free surface. $f \equiv 0$ for all these particles $\forall t$:

$$\boxed{\frac{Df}{Dt} = 0}$$

Example:



use $f(x,y,t) = h(x,t) - y = 0$
if $y = h(x,t)$

Kinematic BC

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u_j \frac{\partial F}{\partial x_j} = 0$$

&

$u_1 = u$

$u_2 = v$ $\quad F = h(x,t) - y$

$x_1 = x$

$x_2 = y$

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow \boxed{\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - v = 0}$$