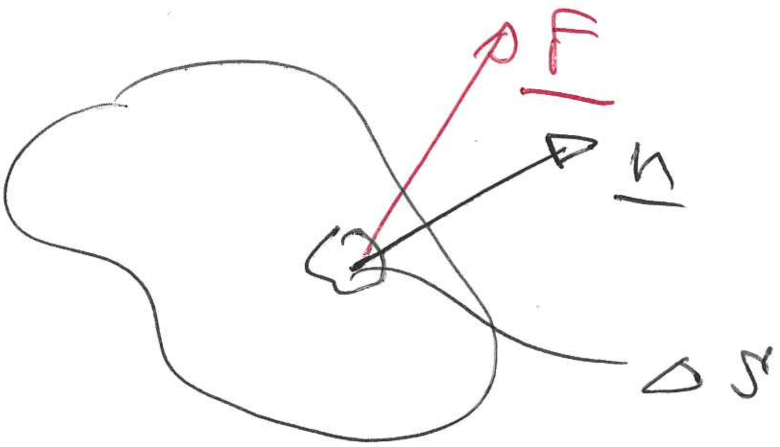
§... Stress, Cauchy's eqn & the Navier-Stokes eqns

① The concept of stress/traction

Consider a finite blob of fluid loaded by some distributed force. (pressure or shear stresses etc)



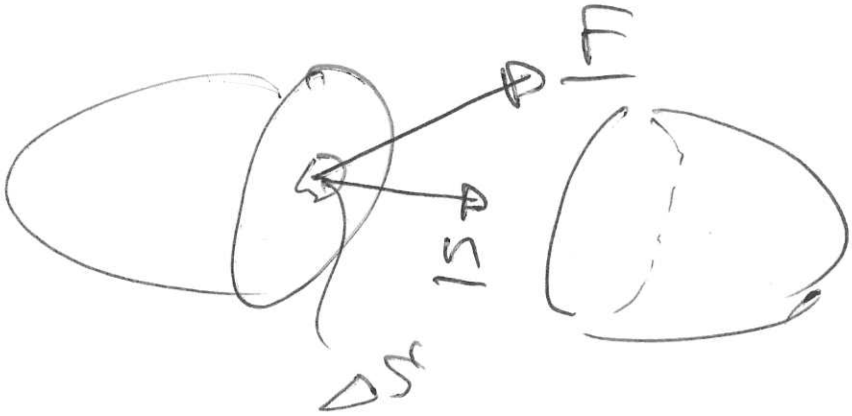
Every patch ΔS on the surface (with local outer unit normal $\underline{n}$) is subject to a certain resulting force $\underline{F}$.

Def: Traction:

$$\boxed{\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S}}$$

acts on the fluid

<u>Similarly:</u> Cut the blob along a plane with outer unit normal $\underline{n}$



Then $\underline{F}$ represents the force exerted onto ΔS (with outer unit normal $\underline{n}$) by the "other half" of the fluid.

<u>Def:</u> Stress

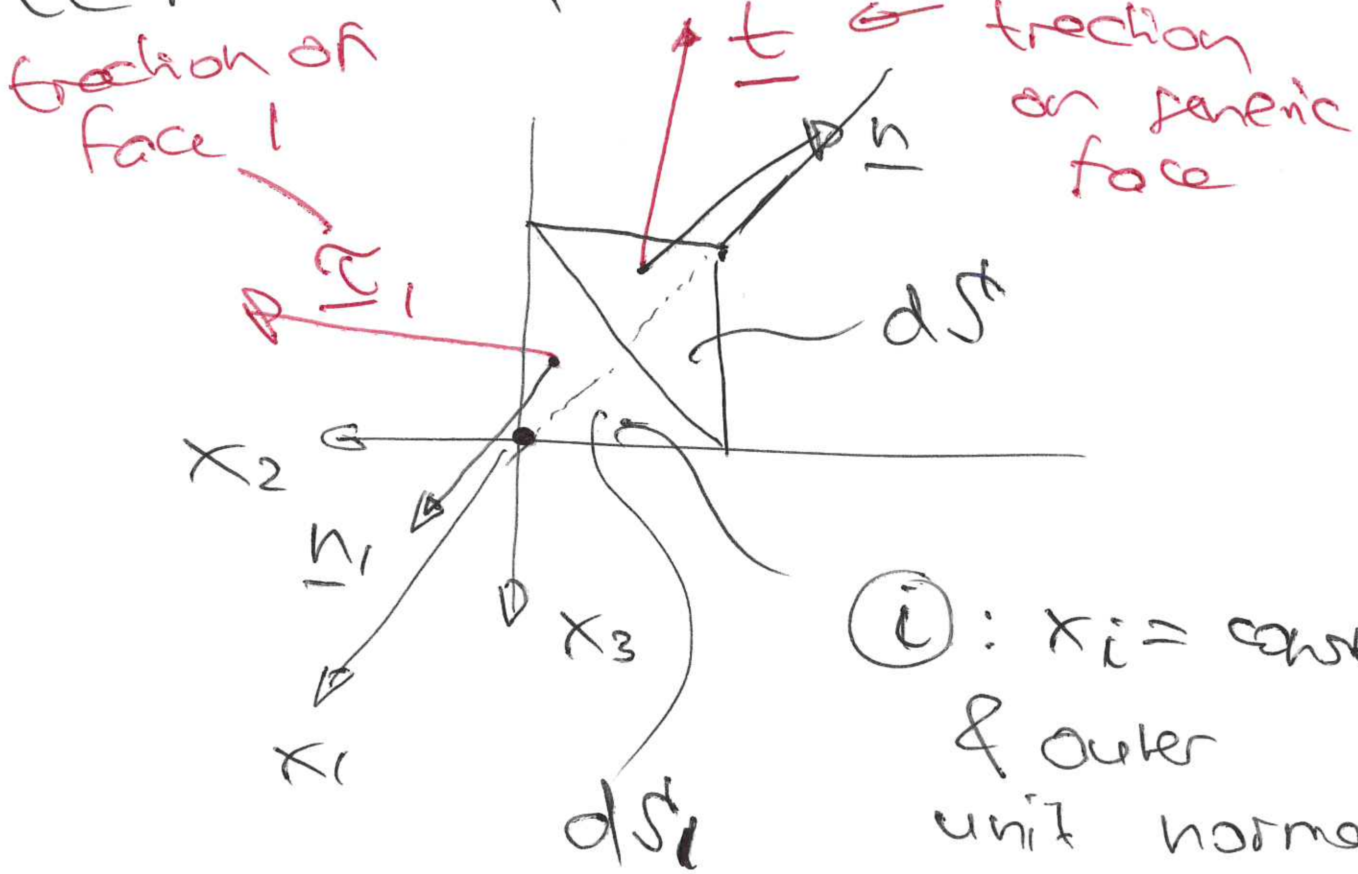
$$\boxed{\underline{t} = \lim_{\Delta S \to 0} \frac{\underline{F}}{\Delta S}}$$

<u>Note:</u> The stress is likely to depend on:

- posn. in fluid
- the direction of $\underline{n}$

② The stress tensor

To examine dependence on $\underline{n}$, consider an infinitesimal tetrahedron



ⓘ : $x_i =$ const & outer unit normal $\underline{n}_i$ points in the pos. x_i-direction

$$\underline{n}_i = \underline{e}_i$$

etc.

(4)

Represent faces (orientation & area) by vectors that are parallel to outer unit normal & whose magnitude is equal to the area.

Then:

$$\underbrace{\underline{n}_i \, dS_i} + \underline{n} \, dS = \underline{0}$$

(Exercise)

$$\cdot \, \underline{e}_j = \underline{n}_j$$

$\underline{e}_i \cdot \underline{e}_j$

$$\underbrace{\underbrace{\underline{n}_i \cdot \underline{n}_j}_{\delta_{ij}} \, dS_i}_{dS_j} + \underbrace{\underline{n} \cdot \underline{n}_j}_{n_j} \, dS = 0$$

$\cdot \, \underline{e}_j$

$$dS_j + n_j \, dS = 0$$

$$\boxed{dS_i = - n_i \, dS}$$

Now: Balance of forces:

Sum of all forces = 0

$$\underline{t}\, dS = -\underline{\tau}_j\, \underbrace{dS_j}_{-n_j\, dS}$$

$$\underline{t}\, \cancel{dS} = \underline{\tau}_j\, n_j\, \cancel{dS}$$

$$\boxed{t_i = \tau_{ij}\, n_j}$$

τ_{ij} = stress tensor

τ_{ij} represents the traction component in the i-th direction on the face where x_j = const. & ~~the~~ whose outer unit normal points in the x_j direction

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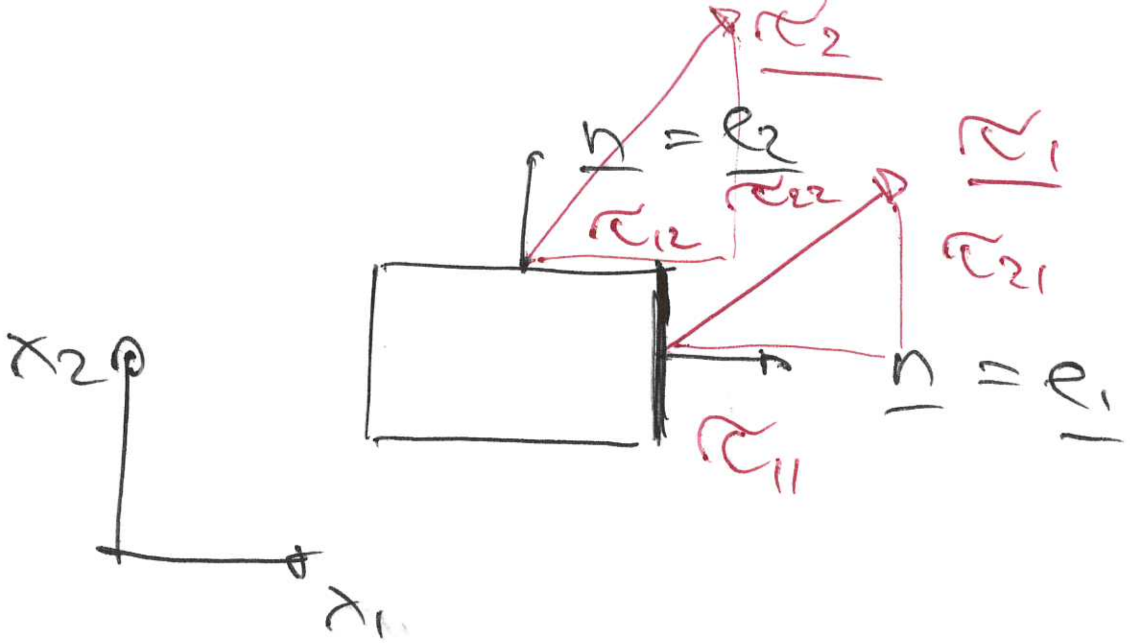

$\underline{\tau_2}$
$\underline{n} = \underline{e_2}$
τ_{22}
τ_{12}
$\underline{\tau_1}$
τ_{21}
$\underline{n} = \underline{e_1}$
τ_{11}
x_2
x_1