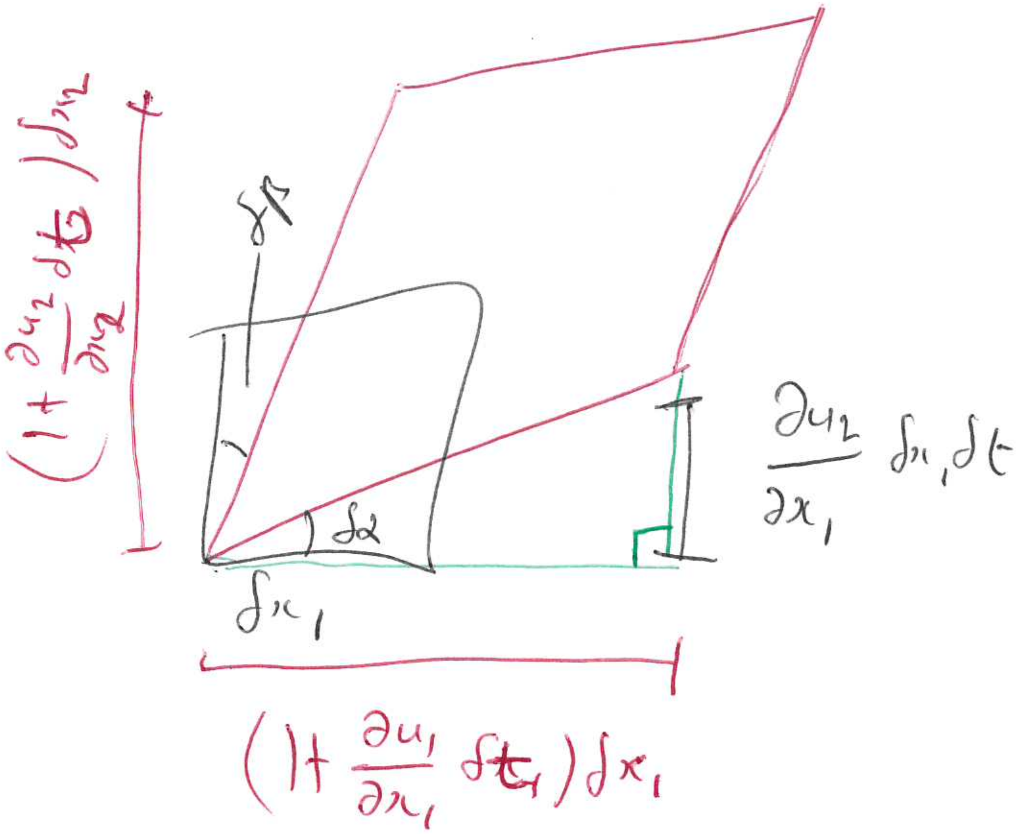
(ii) Shear rate of strain



Change of shape of a small parcel of fluid body with rigid-body motion subtracted

$$\tan \delta\alpha = \frac{\frac{\partial u_2}{\partial x_1}\cancel{\delta x_1}\delta t}{\left(1+\frac{\partial u_1}{\partial x_1}\delta t\right)\cancel{\delta x_1}}$$

Now as $\delta t \to 0$: $\delta\alpha \to 0$

So $\delta\alpha \approx \frac{\partial u_2}{\partial x_1}\delta t$ (neglecting quadratic terms)

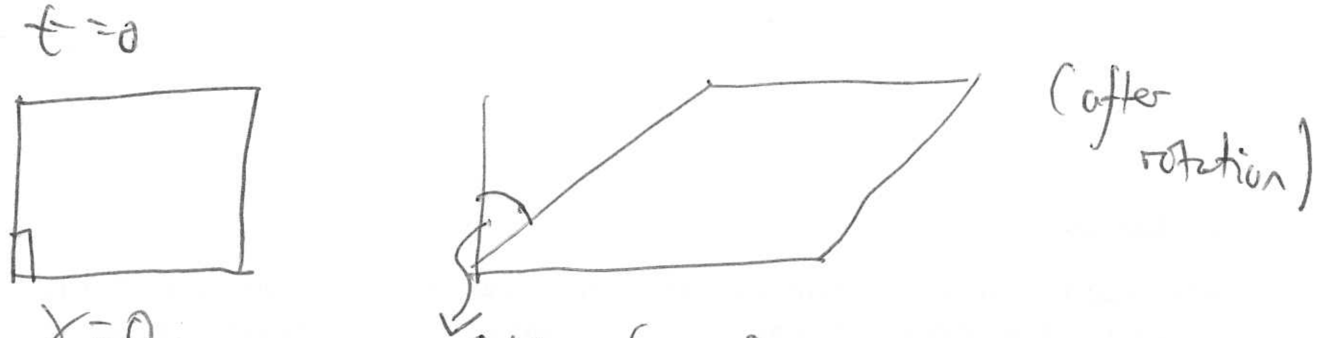
In the limit

$$\boxed{\frac{D\alpha}{Dt} = \frac{\partial u_2}{\partial x_1}}$$ (material line)

Similarly $$\boxed{\frac{D\beta}{Dt} = \frac{\partial u_1}{\partial x_2}}$$

Now consider the "shear rate"

rate at which the right angle gets smaller



$\gamma = 0$

$\delta\gamma = \delta\alpha + \delta\beta$

Thus
$$\frac{D\gamma}{Dt} = \frac{D\alpha}{Dt} + \frac{D\beta}{Dt} = \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = 2\,\varepsilon_{12}$$

The off-diagonal entries in ε_{ij} represent <u>half</u> the shear rate in a plane spanned by $\underset{\sim}{e}_i$ & $\underset{\sim}{e}_j$.

<u>Also</u>

The change in angle of the main diagonal of the fluid rectangle is
$$\delta\phi = \tfrac{1}{2}(\delta\alpha - \delta\beta)$$

So, $\dfrac{D\phi}{Dt} = \dfrac{1}{2}\left(\dfrac{D\alpha}{Dt} - \dfrac{D\beta}{Dt}\right)$

$$= \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \omega_{21} = \omega_3$$

So rotation rate of main diagonal about $\underset{\sim}{e_3}$ is the vorticity, as shown earlier.

Summary

Motion of fluid can be decomposed into

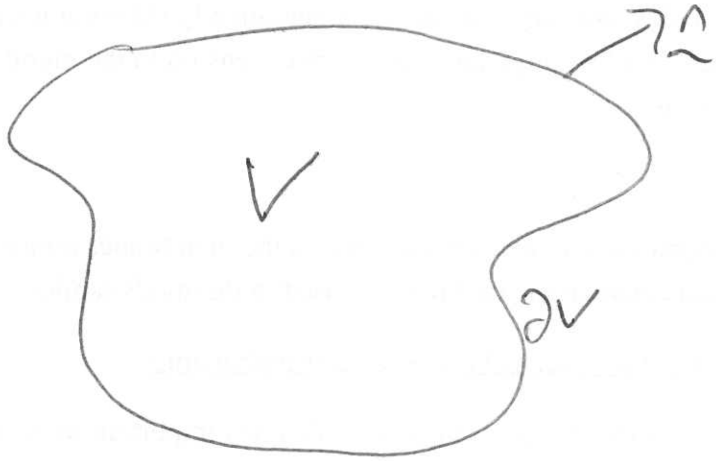
$$\underset{\sim}{u}(\underset{\sim}{x} + \delta\underset{\sim}{x}) = \underbrace{\underbrace{\underset{\sim}{u}(\underset{\sim}{x})}_{\text{translation}} + \underbrace{\underset{\sim}{\Omega} \times \delta\underset{\sim}{x}}_{\text{rotation}}}_{\text{rigid-body.}} + \underbrace{\underset{=}{E}\,\delta\underset{\sim}{x}}_{\text{dilation \& shear}}$$

$$u_i(x_k + \delta x_k) = u_i(x_k) + \omega_{ij}\,\delta x_j + \varepsilon_{ij}\,\delta x_j$$

Equation of continuity

Physics: Mass flux into a spatially fixed control volume
= rate of change of mass in that volume.

Integral form:



Mass flux = density, $\rho \left[\frac{kg}{m^3}\right]$ × velocity normal to hole $\left[\frac{m}{s}\right]$

× cross-sectional area $[m^2]$ = $\left[\frac{kg}{s}\right]$

Rate of change of mass = total influx

$$\frac{d}{dt}\int_V \rho \, dV = \int_{\partial V} -\rho \underline{u} \cdot \underline{n} \, dA$$

↑ negative because $\underline{n}$ is outer unit normal

[volume is fixed ⇒ take $\frac{\partial}{\partial t}$ inside]

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_{\partial V} \rho\, \underline{u} \cdot \underline{n}\, dA = 0$$

Now use divergence theorem

$$\int_V \frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u})\, dV = 0$$

This is true for every possible volume V,

so

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0} \leftarrow \text{continuity equation}$$

In index notation

$$\frac{\partial (\rho u_i)}{\partial x_i} + \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \rho \frac{\partial u_i}{\partial x_i} + \underbrace{u_i \frac{\partial \rho}{\partial x_i} + \frac{\partial \rho}{\partial t}}_{\frac{D\rho}{Dt}} = 0$$

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0}$$

$\frac{D\rho}{Dt}$ is the rate of change of density of material fluid elements.

For an incompressible fluid, $\frac{D\rho}{Dt} = 0$

& then $\boxed{\frac{\partial u_i}{\partial x_i} = 0}$

i.e. $\boxed{\operatorname{div} \underset{\sim}{u} = 0}$

This is "conservation of volume" and is a purely kinematic constraint!