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Flow field comprises:

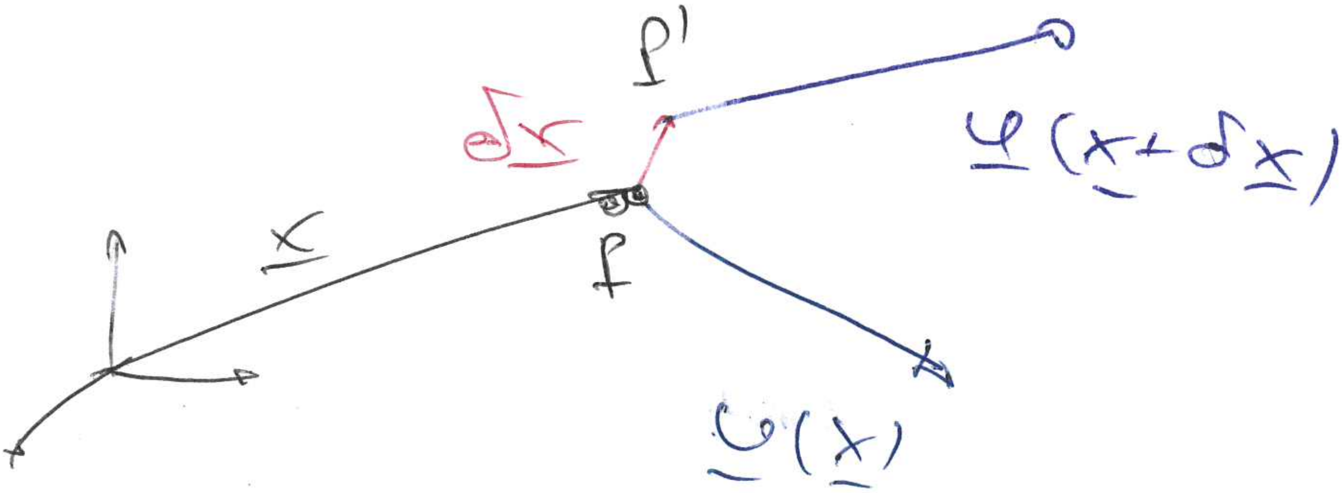
$$\underline{u} = \underline{u}(\underline{x}, t)$$

- Translation
- Rotation

} rigid body

- Dilation
- Shear

} deformation



$$\underline{u}(\underline{x}+\delta\underline{x})-\underline{u}(\underline{x}) = \delta\underline{u}$$

$$\delta u_i = \underbrace{\frac{\partial u_i}{\partial x_j}}\,\delta x_j$$

velocity gradient tensor (3x3 matrix)

If $\delta\underline{u} = \underline{0}$ then all fluid particles translate with the same velocity.

$\Rightarrow$ Rigid body translation happens only if $\frac{\partial u_i}{\partial x_j} = 0$

Other "modes" / "components" of motion must be contained in $\frac{\partial u_i}{\partial x_j}$

To see this split $\frac{\partial u_i}{\partial x_j}$ into its sym. & antisym. parts:

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij}} + \frac{1}{2}\underbrace{\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\omega_{ij}}$$

ε_{ij} rate of strain tensor

ω_{ij} rotation tensor

$$\varepsilon_{ij} = \varepsilon_{ji} \qquad \omega_{ij} = -\omega_{ji}$$

$$\delta u_i = \varepsilon_{ij}\,\delta x_j + \omega_{ij}\,\delta x_j$$

(1) Rigid body rotation / vorticity

Consider the incremental veloc. change caused by ω_{ij}

$$\delta u_i = \omega_{ij}\, \delta x_j$$

matrix vector product
with an antisymm. matrix

$$\begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{pmatrix} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}$$

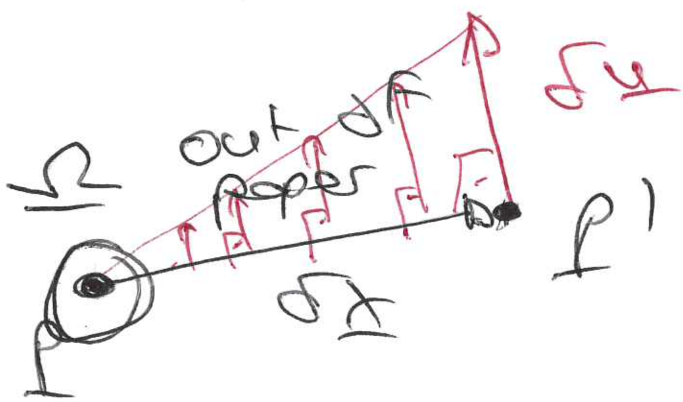
can write this as

$$\delta \underline{u} = \underline{\Omega} \times \delta \underline{x}$$

where

$$\underline{\Omega} = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = \text{rate of rotation vector}$$

Geometrical interpretation



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$\Rightarrow \delta \underline{u} = \underline{\Omega} \times \delta \underline{x}$

creates / corresponds to a rigid body rotation about P with angular veloc $\underline{\Omega}$.

$$\underline{\Omega} = \frac{1}{2}\begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} = \frac{1}{2} \underbrace{\nabla \times \underline{u}}_{\text{curl } \underline{u}}$$

$$\underline{\Omega} = \frac{1}{2}\underline{\omega}$$

vorticity vector.

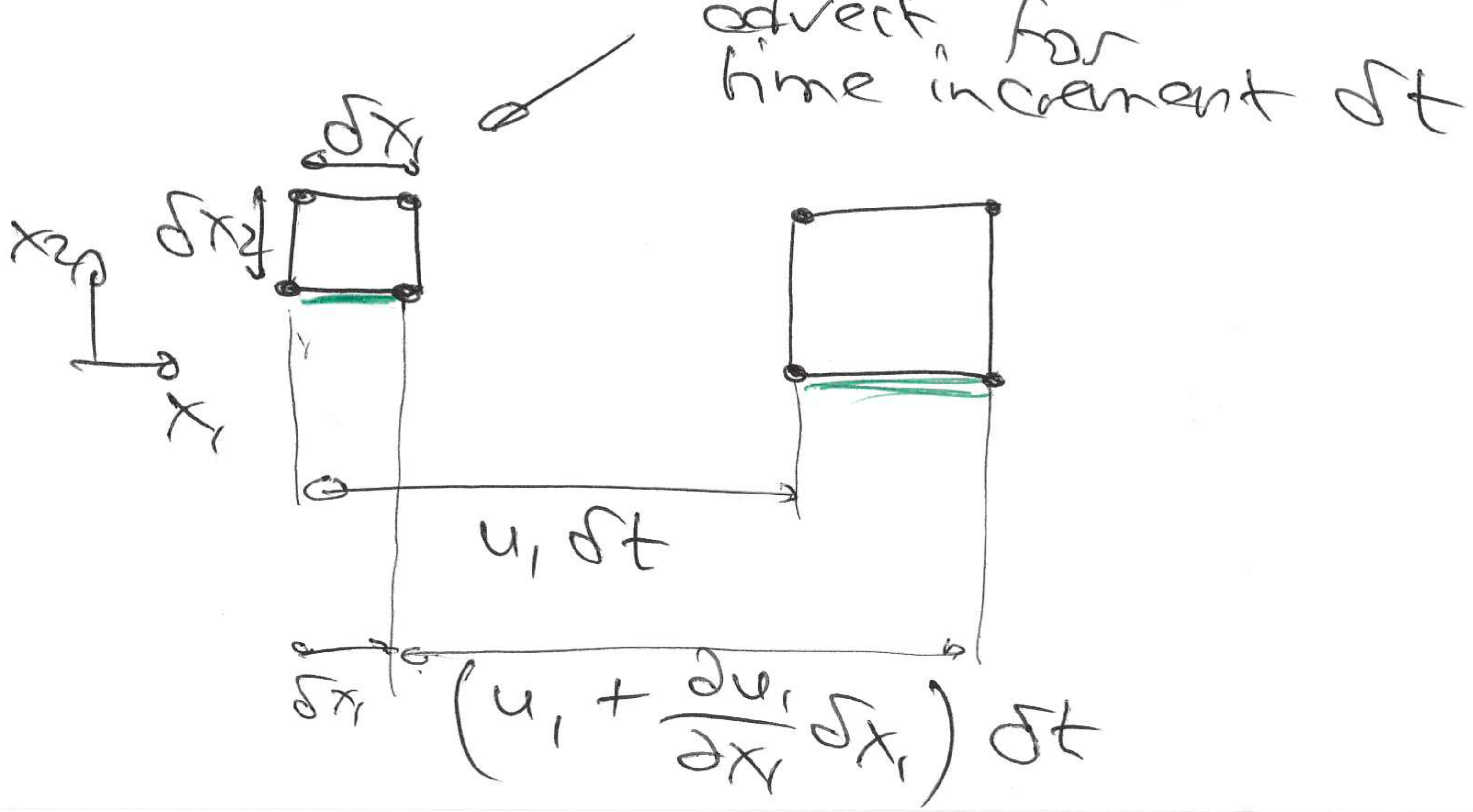
2.1 The rate of strain

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

is the rate of strain tensor & it "contains" deformation of fluid particles: Dilation & shear.

(i) Extensional rate of strain

~~Ill~~ Illustration (2D)



δx_1

$$\text{Strain} = \frac{\text{length} - \text{old length}}{\text{old length}}$$

δx_{11}

$$= \frac{\left\{\left[\cancel{\delta x_1} + \left(\cancel{u_1} + \frac{\partial u_1}{\partial x_1}\delta x_1\right)\delta t\right] - \cancel{u_1 \delta t}\right\} - \cancel{\delta x_1}}{\delta x_1}$$

$$\text{strain} = \frac{\partial u_1}{\partial x_1}\delta t$$

$$\text{rate of strain} = \frac{\partial(\text{strain})}{\partial t}$$

$$= \frac{\partial u_1}{\partial x_1} = \varepsilon_{11}$$

Similar for other directions: ε_{11} (etc) (the diag. elems of e_{ij}) represent the extensional rate of strain in the direction of the 3 coord axes..