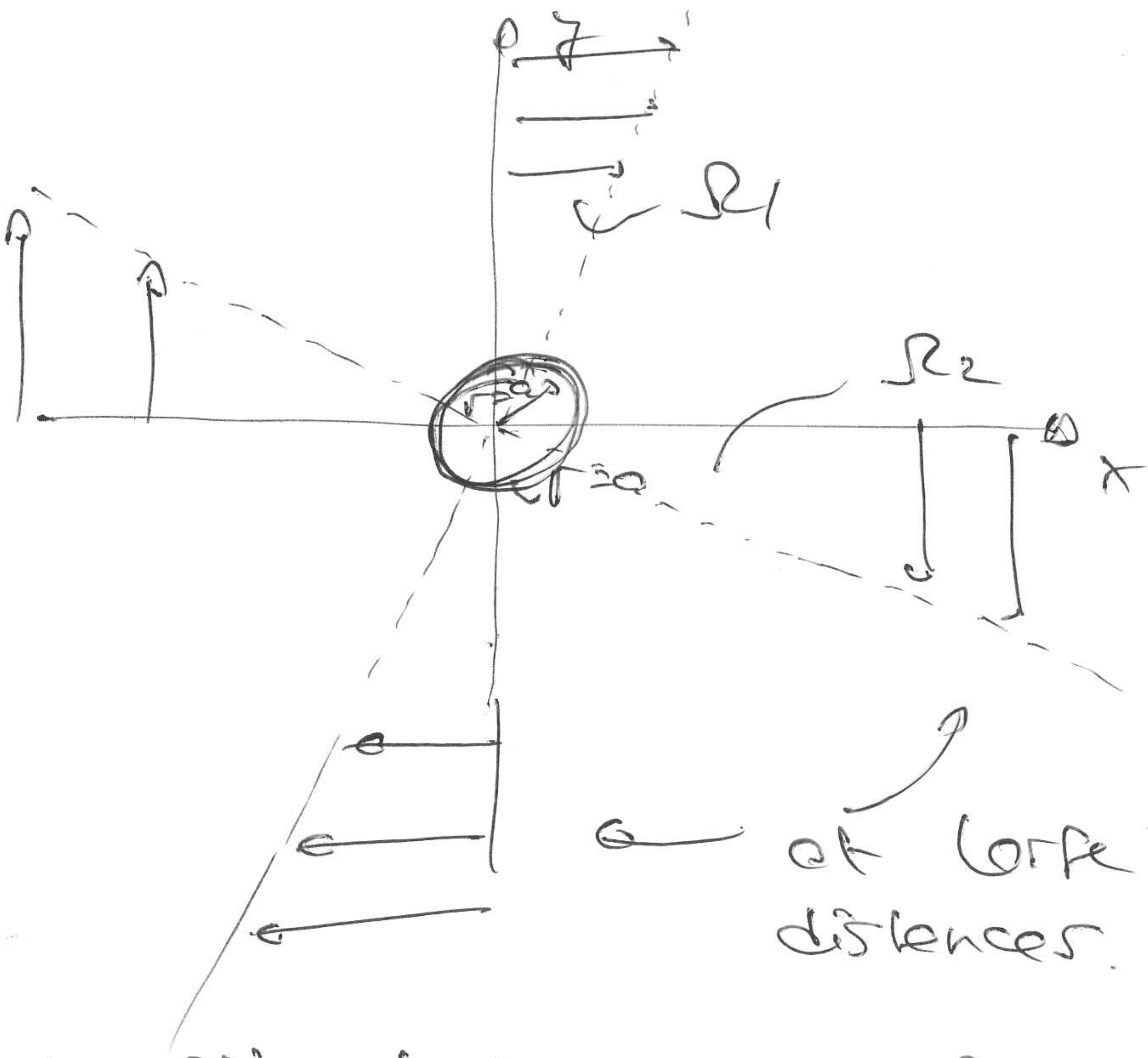
Example: Cylinder in double shear flow



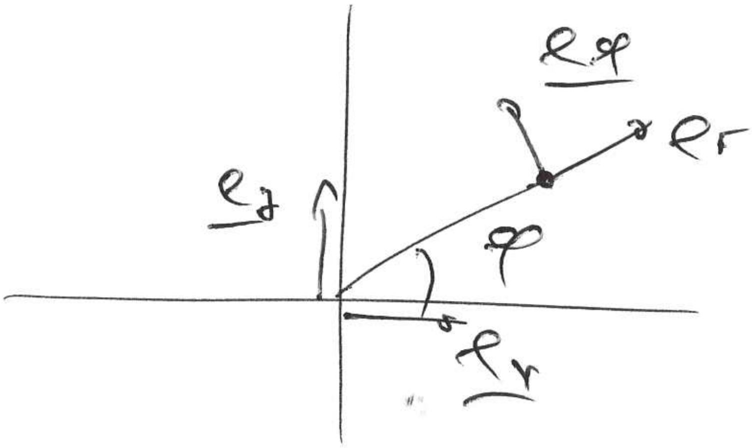
- no slip at $r=a$: $\underline{u} = \underline{0}$
- in far field as $r \to \infty$

$$\underline{u} \to \Omega_1 y \, \underline{e_x} - \Omega_2 x \, \underline{e_y}$$

Polars: $x = r\cos\varphi$
$y = r\sin\varphi$

$$\underline{e_x} = \underline{e_r}\cos\varphi - \underline{e_\varphi}\sin\varphi$$

$$\underline{e_y} = \underline{e_r}\sin\varphi + \underline{e_\varphi}\cos\varphi$$



So as $r \to \infty$:

$$\underline{u} \to \Omega_1 r\sin\varphi\left(\underline{e_r}\cos\varphi - \underline{e_\varphi}\sin\varphi\right) - \Omega_2 r\cos\varphi\left(\underline{e_r}\sin\varphi + \underline{e_\varphi}\cos\varphi\right)$$

$$\to \underline{e_r}\Big[r\underbrace{\sin\varphi\cos\varphi}_{\frac{1}{2}\sin(2\varphi)}(\Omega_1-\Omega_2)\Big] + \underline{e_\varphi} r\Big[-\Omega_1\underbrace{\sin^2\varphi}_{\frac{1}{2}(1-\cos(2\varphi))} - \Omega_2\underbrace{\cos^2\varphi}_{\frac{1}{2}(1+\cos(2\varphi))}\Big]$$

$$\underline{u} \sim u_r\underline{e_r} + u_\varphi\underline{e_\varphi}$$

As $r \to \infty$

$$u_r \to \frac{1}{2} r\sin(2\varphi)(\Omega_1-\Omega_2)$$

$$u_\varphi \to -\frac{1}{2} r\left[(\Omega_1+\Omega_2) + (\Omega_2-\Omega_1)\cos(2\varphi)\right]$$

Try stream fct: $D^4\psi = 0$ 3

$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi}$ $u_\varphi = -\frac{\partial \psi}{\partial r}$

Assume $\psi \sim \cos 2\varphi$
will satisfy the φ dependence of the φ-dependent terms in both BCs

But assuming $\psi(r,\varphi) = f(r)\cos(2\varphi)$ would make all terms in veloc vary with φ $\Rightarrow$ not consistent with u_φ BC.

This suggests ansatz:

$\psi(r,\varphi) = F(r) + G(r)\cos(2\varphi)$

Note ψ has to satisfy $D^4\psi = 0$; the general soln of this biharmonic eqn in (r,φ) is known.

$\Rightarrow$ Handout.

8.2.1 The streamfunction and the biharmonic equation in cylindrical polars

- In cylindrical polars, (r, φ) the relation between the streamfunction $\psi(r, \varphi)$ and the velocity components u_r and u_φ is:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \tag{8.14}$$

and

$$u_\varphi = -\frac{\partial \psi}{\partial r}, \tag{8.15}$$

where $\mathbf{u} = u_r \mathbf{e}_r + u_\varphi \mathbf{e}_\varphi$.

- The biharmonic equation in polar coordinates:

$$\nabla^4 \psi(r, \varphi) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right]\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi^2}\right] \tag{8.16}$$

$$\nabla^4 \psi(r, \varphi) = \psi_{,rrrr} + \frac{2}{r}\psi_{,rrr} - \frac{1}{r^2}(\psi_{,rr} - 2\psi_{,rr\varphi\varphi}) + \frac{1}{r^3}(\psi_{,r} - 2\psi_{,r\varphi\varphi}) + \frac{1}{r^4}(4\psi_{,\varphi\varphi} + 2\psi_{,\varphi\varphi\varphi\varphi}) \tag{8.17}$$

- For axisymmetric solutions:

$$\nabla^4 \psi(r) = \frac{1}{r}\left[r\left(\frac{1}{r}[r\psi_{,r}]_{,r}\right)_{,r}\right]_{,r} \tag{8.18}$$

$$\nabla^4 \psi(r) = \psi_{,rrrr} + \frac{2}{r}\psi_{,rrr} - \frac{1}{r^2}\psi_{,rr} + \frac{1}{r^3}\psi_{,r} \tag{8.19}$$

- The general form of the solution of the biharmonic equation in cylindrical polars is known. It can be represented by superposition of the following solutions:
  - The general axisymmetric solution:

$$\psi(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \tag{8.20}$$

$f_0(r)$

  - The general separated non-axisymmetric solution:

For $n = 1$:

$$\psi(r, \varphi) = \left(Ar + \frac{B}{r} + Cr^3 + Dr \log r\right) \cos(\varphi) + \left(ar + \frac{b}{r} + cr^3 + dr \log r\right) \sin(\varphi) \tag{8.21}$$

For $n \geq 2$:

$$\psi(r, \varphi) = \sum_{n=2}^{\infty} \left(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}\right) \cos(n\varphi) + \left(a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}\right) \sin(n\varphi) \tag{8.22}$$

The coefficients $(A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1, a_1, b_1, c_1, d_1, A_2, B_2, C_2, D_2, a_2, b_2, c_2, d_2, \ldots)$ have to be determined from the boundary conditions.

$\Rightarrow$ Choose $F(r)$ & $G(r)$ from the known form of the soln. : $\psi = F(r) + G(r)\cos(2\varphi)$

$$\psi = \underbrace{\left(A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r\right)}_{F(r)} +$$

$$+ \underbrace{\left(A_2 r^2 + B_2 r^{-2} + C_2 r^4 + D_2\right)}_{G(r)} \cos(2\varphi)$$

Note: This satisfies the biharmonic eqn (by construction!) & has the desired functional form.