$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

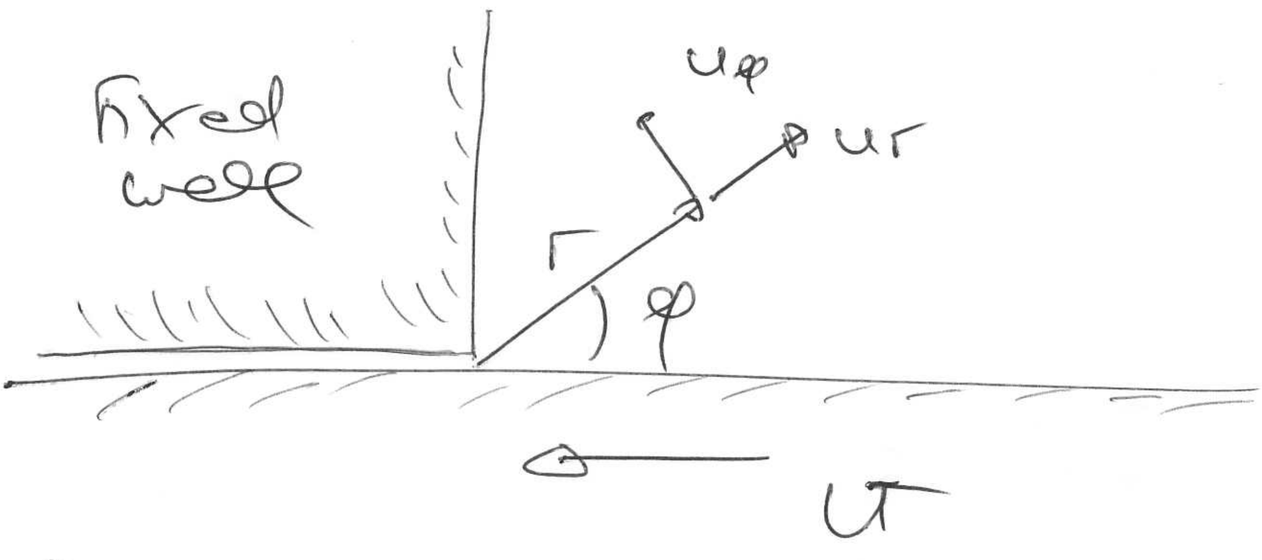
satisfies continuity

2D Stokes; $Re = \frac{U a \rho}{\mu} \ll 1$

$$\nabla^4 \psi = 0$$

Example for 2D Stokes flow

Scraping flow



Assume: slow steady viscous flow so that $Re \ll 1$

$$\nabla^4 \psi = 0$$

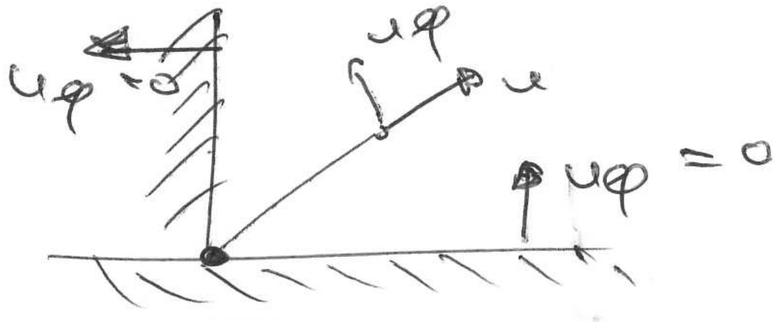
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Streamfct in cyl. polars:

$$u = u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi}$$

$$v = u_\varphi = -\frac{\partial \psi}{\partial r}$$

BC: Impermeability:



$$v = u_\varphi = \underbrace{-\frac{\partial \psi}{\partial r} = 0}\quad \text{at } \varphi = 0 \quad \underbrace{\forall r}$$

$$\Rightarrow \psi = \text{const} = C_1 \quad \text{at } \varphi = 0$$

$$v = u_\varphi = \underbrace{-\frac{\partial \psi}{\partial r} = 0}\quad \text{at } \varphi = \frac{\pi}{2} \quad \underbrace{\forall r}$$

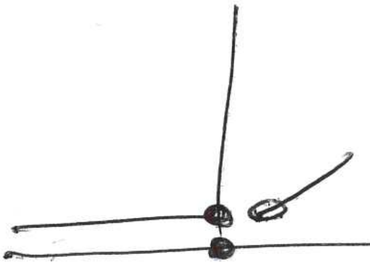
$$\psi = \text{const} = C_2 \quad \text{at } \varphi = \pi/2$$

Continuity of streamfct where 2 walls meet:

$$C_1 = C_2 = C = \text{arb. constant}$$

set it to zero.

If $C_1 \neq C_2$:

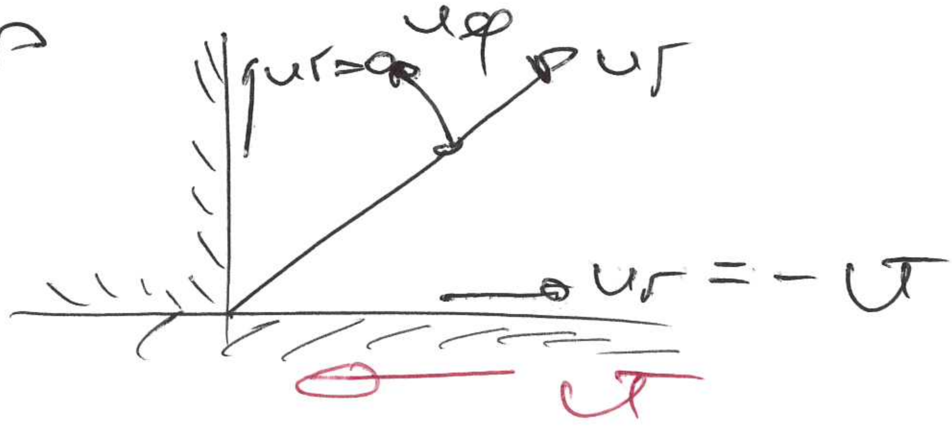


jump in streamfct

$\Rightarrow$ flux into (nonexistent) gap.

$$\boxed{\begin{aligned} \psi(\varphi = 0) &= 0 \quad (1) \\ \psi\left(\varphi = \tfrac{\pi}{2}\right) &= 0 \quad (2) \end{aligned}}$$

No slip



$$\boxed{\begin{aligned} u &= 0 = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad \text{at } \varphi = \frac{\pi}{2} \qquad (3)\\ u &= -U = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad \text{at } \varphi = 0 \qquad (4)\end{aligned}}$$

4

$$\boxed{\nabla^4 \psi = 0}$$

<u>Ansatz:</u> <u>Try</u> separated soln

$$\psi(r,\varphi) = g(r)\, f(\varphi)$$

<u>Note:</u> BCs (3) & (4) show that $\frac{1}{r}\frac{\partial \psi}{\partial \varphi}$ should be indep of r (for 2 particular values of φ)

<u>Try:</u> $g(r) = Ur$

This allows (3) & (4) to be satisfied; <u>hopefully</u> $f(\varphi)$ can accommodate (1) & (2).

$$\psi(r,\varphi) = Ur\, f(\varphi)$$

into BCs:

$\boxed{5}$

$\varphi = 0: \ \psi = 0 \Rightarrow f(0) = 0$

$\varphi = \frac{\pi}{2}: \ \psi = 0 \Rightarrow f(\frac{\pi}{2}) = 0$

$\varphi = 0: \ \frac{1}{r}\frac{\partial \psi}{\partial \varphi} = -U \Rightarrow f'(0) = -1$

$\varphi = \frac{\pi}{2}: \ \frac{\partial \psi}{\partial \varphi} = 0 \rightarrow f'(\frac{\pi}{2}) = 0$

Ansatz into PDE

$$\nabla^4 \psi = \nabla^2 \nabla^2 \psi = 0$$

$$\nabla^2 \psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) U r f(\varphi)$$

$$\frac{U}{r} f(\varphi) + \frac{U}{r} f''(\varphi) = U r^{-1}(f + f'')$$

$$\nabla^2 \nabla^2 \psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) U r^{-1} (f + f'')$$

$$= U \Big\{ (-1)(-2) r^{-3} (f + f'') + (-1) r^{-3} (f + f'') + r^{-3} (f'' + f^{IV}) \Big\}$$

$$\nabla^4 \psi = \frac{U}{r^3}\left((2-1) f + (2 - 1 + 1) f'' + f^{IV}\right)$$

$$\nabla^4 \varphi = \frac{U}{r^3} \underbrace{\left(f + 2f'' + f^{IV} \right)}_{=0} = 0$$ (6

4^{th} order lin const. coeff'n ODE for $f(\varphi)$.

$$f(\varphi) \sim e^{\lambda \varphi}$$

Char. poly:

$$1 + 2\lambda^2 + \lambda^4 = 0$$

$$(\lambda^2 + 1)^2 = 0$$

$$\lambda_{1234} = \pm i$$

$$f(\varphi) = A \sin\varphi + B \cos\varphi + C \varphi \sin\varphi + D \varphi \cos\varphi$$

because of repeated root.

Now apply 4 BCs.

$\vdots$

$$\psi(r,\varphi) = \frac{Ur}{\left(\frac{\pi}{2}\right)^2 - 1}\left(-\left(\frac{\pi}{2}\right)^2 \sin\varphi + \varphi\cos\varphi + \frac{\pi}{2}\varphi\sin\varphi\right).$$