§.. Stream fct & vorticity eqns

Alternative formulation of the N.St. eqns, powerful in 2D.

[mainly in this section: 2D & incompressible]

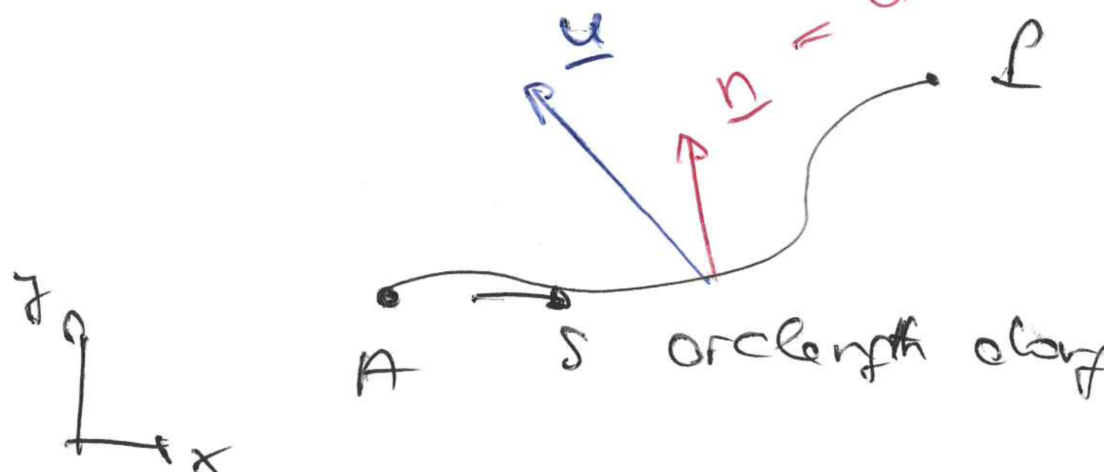
Stream fct:

$$\underline{u} = u\,\underline{e_x} + v\,\underline{e_y}$$

Def:

$$\psi_A(P) = \int_A^P \underline{u}\cdot\underline{n}\, ds$$

$\underline{n}$ = unit normal to path & to its left.



s arclength along path.

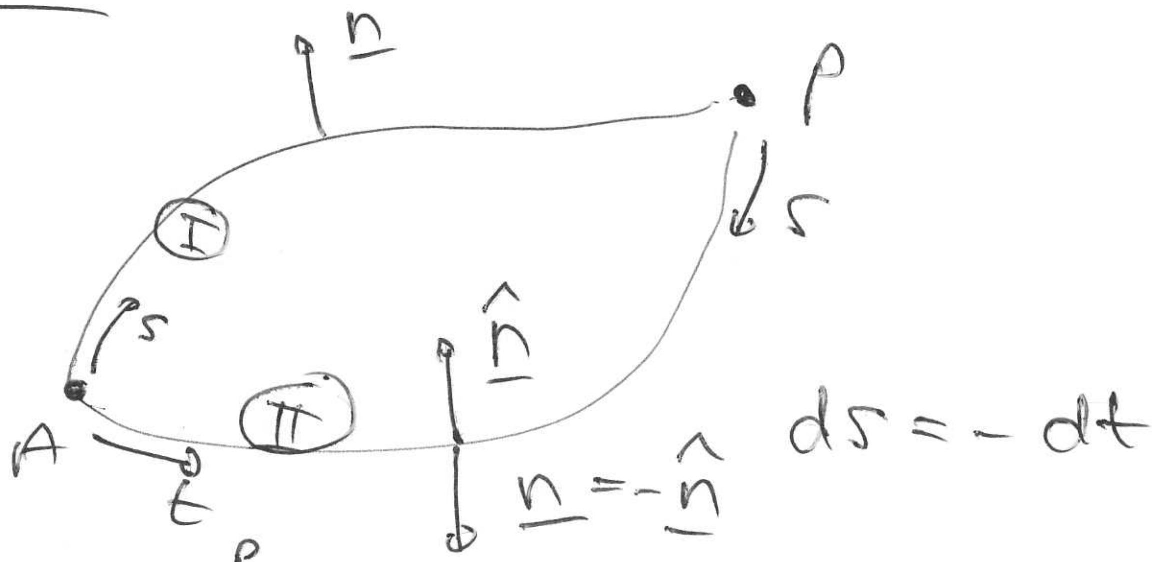
$\underline{u}\cdot\underline{n}$ is veloc comp that crosses the path.

$\psi_A(P)$ is the volume flux per unit depth in the z-direction crossing the line AP

Implications:

(1) $\psi_A(P)$ is path indep!

Proof:



$$\psi_A^{I}(P) = \int_{(I)A}^{P} \underline{u}\cdot\underline{n}\, ds$$

$$\psi_A^{(II)}(P) = \int_{(II)A}^{P} \underline{u}\cdot\underbrace{\hat{\underline{n}}}_{-\underline{n}}\, \underbrace{dt}_{-ds}$$

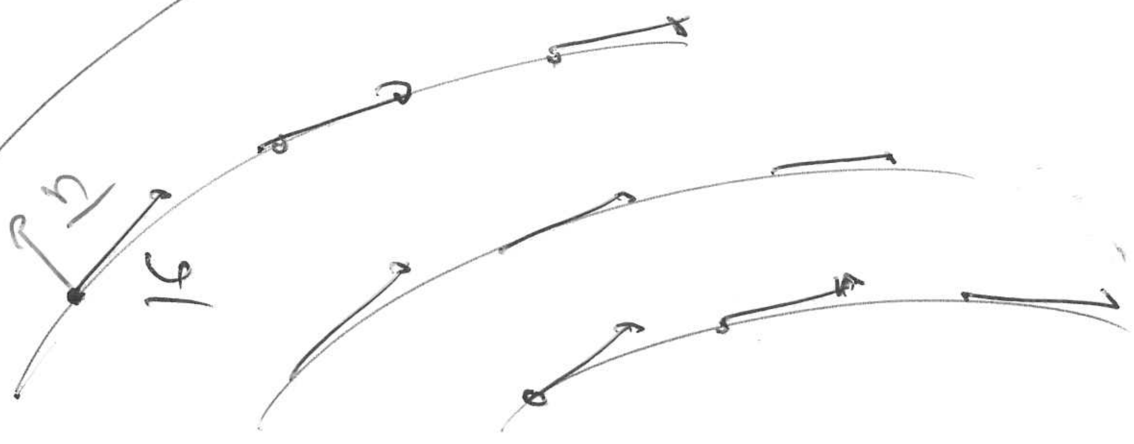
$$= -\int_{(II)P}^{A} \underline{u}\cdot\underline{n}\, ds$$

$$\psi_A^{\mathrm{I}}(P) - \psi_A^{\mathrm{II}}(P) = \int_{\mathrm{I}\,A}^{P} \underline{u}\cdot\underline{n}\,ds + \int_{\mathrm{II}\,P}^{A} \underline{u}\cdot\underline{n}\,ds$$

$$= \oint_{APA} \underline{u}\cdot\underline{n}\,ds = 0$$

because of incompress.
(int. form of conti eqn). q.e.d.

(2) ψ is constant along streamlines (lines that are tangential to veloc)



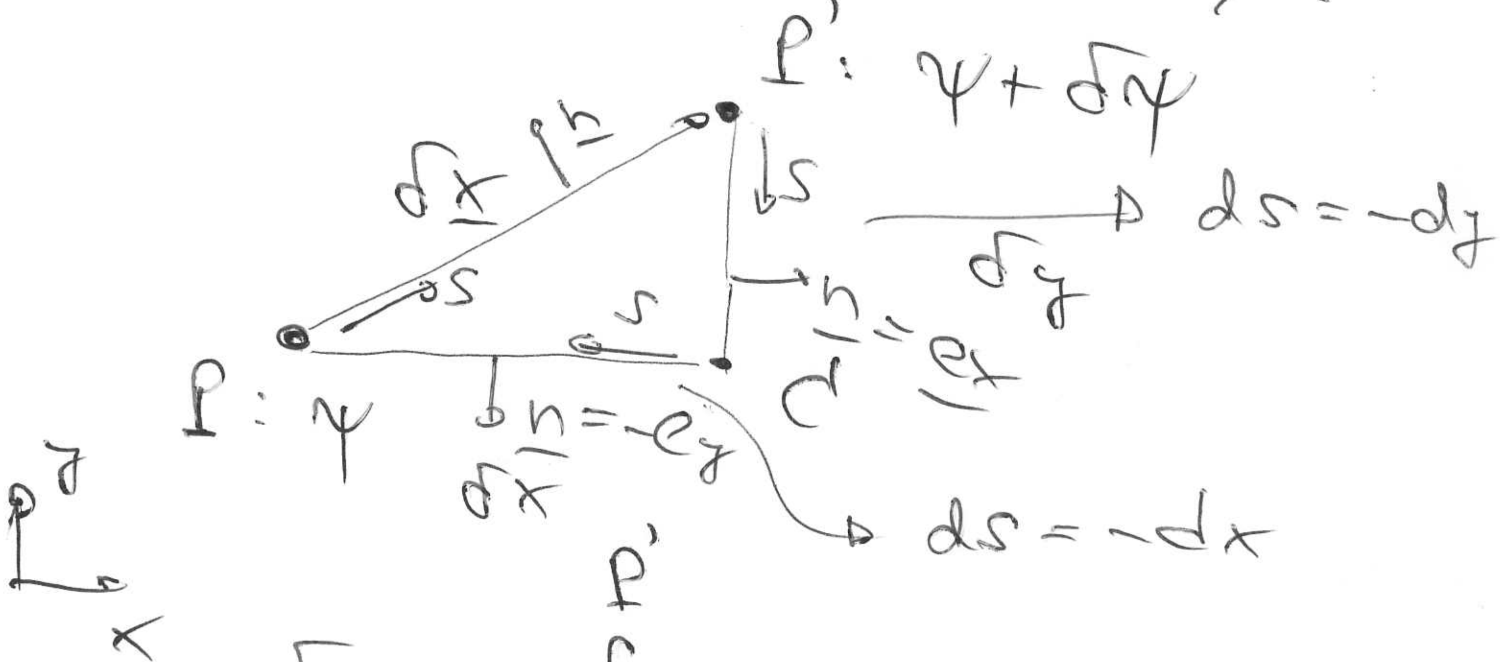
along these lines

$$\underline{u}\cdot\underline{n} = 0$$

(3) impermeable boundaries are streamlines in this sense.

<u>Convention:</u> Set $\psi = 0$ along solid, stationary boundaries (if possible).

What PDE does ψ satisfy?



$$\underbrace{\delta\psi}_{\psi_P(P')} = \int_P^{P'} \underline{u}\cdot\underline{n}\, ds$$

Continuity: $\oint_{\triangle} \underline{u}\cdot\underline{n}\, ds = 0$

$$\delta\psi = \int_P^{P'} \underline{u}\cdot\underline{n}\, ds = -\int_{P'}^{C} \underbrace{\underline{u}\cdot\underbrace{\underline{n}}_{e_x}}_{u}\, \underbrace{ds}_{(-dy)} - \int_{C}^{P} \underbrace{\underline{u}\cdot\underbrace{\underline{n}}_{-e_y}}_{-v}\, \underbrace{ds}_{-dx}$$

$$\delta\psi = +\int_{P'}^{C} u\,dy - \int_{C}^{P} v\,dx$$

Mean value theorem:

$$\boxed{\delta\psi = +u\,\delta y - v\,\delta x} + \ldots$$

as $\delta x, \delta y, \delta\psi \to 0$

Also: $\psi(x,y)$

$$\boxed{\delta\psi = \frac{\partial\psi}{\partial x}\,\delta x + \frac{\partial\psi}{\partial y}\,\delta y} + \ldots$$

$$\boxed{\begin{aligned} \frac{\partial\psi}{\partial x} &= -v \\ \frac{\partial\psi}{\partial y} &= u \end{aligned}}$$

Similar to:
- potential
- Airy stress fct.
- Cauchy Riemann

Remarks:

(1) Derivation involved continuity eqn.
$\Longrightarrow$ continuity eqn should automatically be satisfied.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

$\Rightarrow$ can ignore continuity eqn.

(2) Relation to vorticity (2D)

$$\underline{\omega} = \omega_3 \underline{e_z} = \omega \underline{e_z}$$

$$\left(\underline{\omega} = \nabla \times \underline{u}\right)$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\boxed{\omega = -\nabla^2 \psi}$$