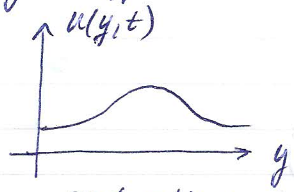
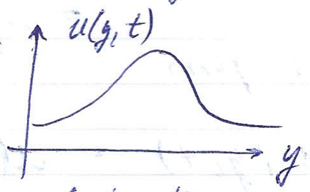
Lecture 15 | WB 13/11/2013

Similarity Solutions 'SS'

Often solutions are self-similar; i.e. they have the same „slope", but possibly a different scale at different times. E.g.:



early-time



later time

The generic form of such solution is

$$\boxed{u(y;t) = a(t)\, f\left(\frac{y}{b(t)}\right)}$$

$a(t)$ scales the amplitude
$b(t)$ scales the width of f

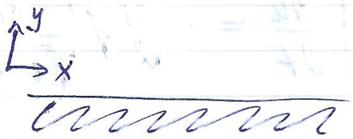
Similarity Solutions do NOT always exist! Try it...

- often SS reduces PDE's to ODE's for $f(\mu)$ where $\mu = \dfrac{y}{b(t)}$
- the existence of SS is often motivated by dimensional arguments

(Ex) Rayleigh's Jerked Plate

Parallel Flow: $\underline{u} = u(y;t)\,\underline{e}_x$

Governing Eqs: $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2}$



$\vec{U}$ at $t>0$

ICs: $u(t=0) = 0$

BCs: $u(y=0) = U$ for $t>0$! not homogeneous!

$u \to 0$ as $y \to \infty$

Observations:

- PDE is linear! $f(\alpha x) = \alpha f(x)$
- All eqs are homogeneous (input $u=0$ & it should work!) apart from 1^{ST} BC!
- Solution will depend on $\underline{u = u(y,t,\nu,U) = U f(y,t,\nu)}$
  (from linearity & homogeneousness of eq. used assuming uniqueness)
  - linearity $\Rightarrow$ any linear combination of u is also a solution
  - nonhomogenous 1^{ST} BC $\Rightarrow$ $f \neq f(U)$
  - dimension of U is length/time is the same as u
    - $\Rightarrow$ $f(y,t,\nu)$ is DIMENSIONLESS fcn of its argument
    - $\Rightarrow$ y, t & ν must combine to sth. dimensionless

$[y] = m$

$[t] = s$

$$[\nu] = \frac{[\partial u/\partial t]}{[\partial^2 u/\partial y^2]} = \frac{m/s^2}{1/ms} = \frac{m^2 s}{s^2} = \frac{m^2}{s}$$

- Dimensionless combination is $\eta = \dfrac{\nu t}{y^2}$ or $\eta = \dfrac{y}{\sqrt{\nu t}}$

The latter choice is the only one that ~~requires~~ gives required similarity form.

$$u(y,t,\nu,U) = Uf(y,t,\nu)$$
$$= Uf\left(\frac{y}{\sqrt{\nu t}}\right)$$

Compare to the form

$$u(y,t) = a(t)f\left(\frac{y}{b(t)}\right)$$

One tends to try to keep the similarity variable η linear in y...

ANSETZ: $\quad u(y,t,\nu,U) = U f(\underbrace{y(\nu t)^{-1/2}}_{\eta = y(\nu t)^{-1/2}})$

into PDE: $\boxed{\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2}}$

- $\dfrac{\partial u}{\partial t} = U \dfrac{df}{d\eta}\dfrac{\partial \eta}{\partial t}$

$$= Uf' y\left(-\tfrac{1}{2}\right)\nu^{-1/2} t^{-3/2}$$

$$\left| \frac{\partial u}{\partial t} = -\frac{U}{2} f' \frac{y}{\sqrt{\nu t^3}} \right|$$

- $\dfrac{\partial u}{\partial y} = U \dfrac{df}{d\eta}\dfrac{\partial \eta}{\partial y}$

$$\frac{\partial u}{\partial y} = Uf' \frac{1}{\sqrt{\nu t}}$$

$$\left| \frac{\partial^2 u}{\partial y^2} = Uf'' \frac{1}{\nu t} \right|$$

then $\quad \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2} \Rightarrow -\dfrac{\cancel{U}}{2} f' \dfrac{y}{\sqrt{\nu t^3}} = \cancel{\nu}\cancel{U} f'' \dfrac{1}{\cancel{\nu} t}$

$$\therefore \boxed{f'' + \frac{1}{2}\eta f' = 0}$$

ODE not PDE!

BC_s: $u(y,t) = Uf\left(\underbrace{\dfrac{y}{\sqrt{\nu t}}}_{\eta}\right) = Uf(\eta)$

- $u(y=0) = U$
  $Uf(0) = U$
  $\underline{f(0) = 1}$
- $u \to 0$ as $y \to \infty$
  $\underline{Uf(\eta) \to 0 \text{ as } \eta \to \infty}$

IC_s: $u(t=0) = 0$
$u \to 0$ as $t \to \infty$
$\underline{f(\eta) \to 0 \text{ as } \eta \to \infty}$

same! 3 $cond_s$ for 2^{ND} order ODE
$\Rightarrow$ PROBLEM

So $IC \equiv 2^{ND} BC \Rightarrow$ 2 $cond_s$ for 2^{ND} order ODE!

ODE: $f'' + \frac{1}{2}\eta f' = 0$

BCs: $f(\eta = 0) = 1$
$f \to 0$ as $\eta \to \infty$

ICs: $f = 0$ as $\eta \to \infty$

Substitute $f' = F \Rightarrow F' + \frac{1}{2}\eta F = 0$

Separation of variables $\frac{F'}{F} = -\frac{1}{2}\eta$

$$\ln\left(\frac{F}{F_0}\right) = -\frac{1}{4}\eta^2 \Rightarrow F = F_0 \exp\left(-\frac{1}{4}\eta^2\right)$$

Sub. back: $F = f' \Rightarrow f = A + F_0 \int^{\eta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi = f(\eta)$

Lower limit is arbitrary; we chose ∞.
So

$$f(\eta) = A + F_0 \int_{\infty}^{\eta} \exp\left(-\frac{1}{4}\xi^2\right) d\xi \qquad -B = F_0$$

$$f(\eta) = A + B \int_{\eta}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi$$

BCs: $f(\eta \to \infty) = 0 = A$
$f(\eta = 0) = 1 = B \int_0^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi = B\sqrt{\pi}$

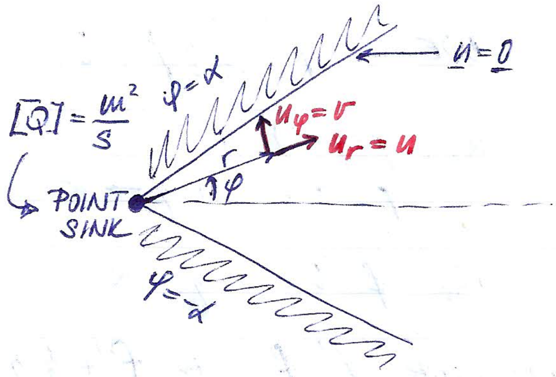
Then

$$\boxed{u = \frac{U}{\sqrt{\pi}} \int_{\eta}^{\infty} \exp\left(-\frac{1}{4}\xi^2\right) d\xi} \quad \text{where} \quad \boxed{\eta = \frac{y}{\sqrt{\nu t}}}$$

Ex Jeffrey - Hamel Flow

Flow: $\underline{u} = \underline{u}(r, \varphi, \rho, \nu, \alpha, Q)$

$[Q] = \frac{m^2}{s}$



Dimensions: $[\rho] = kg/m^3$
$[\nu] = m^2/sec$
$[Q] = m^2/sec$
$[\alpha] =$ dimensionless
$[\varphi] =$ dimensionless
$[r] = m$
$[\underline{u}] = m/sec$

For dimensional consistency: $\underline{u} = \frac{\nu}{r} \underline{f}(r, \varphi, \rho, \nu, \alpha, Q)$

Q would worked too!

must be NON-DIMENSIONAL of its argument!

$\underline{f}$ - cannot depend on ρ (cannot feature kg)
- $\not\propto r$ (nothing to balance m)
- ν & Q can only appear as $\frac{\nu}{Q}$ or $\frac{Q}{\nu}$

$\therefore \quad \underline{f}\left(\varphi, \alpha, \frac{Q}{\nu}\right)$

$$\therefore \quad \boxed{\underline{u} = \frac{\nu}{r} \underline{f}\left(\varphi, \alpha, \frac{Q}{\nu}\right)}$$